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$\boldsymbol{HF} = \boldsymbol{HM}$ V: Seiberg-Witten-Floer homology and handle additions

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This is the last of five papers [KLT1, KLT2, KLT3, KLT4] that construct an isomorphism between the Seiberg-Witten Floer homology and the Heegaard Floer homology of a given compact, oriented 3-manifold. See Theorem 1.4 below for a precise statement. As outlined in [KLT1], this isomorphism is given as a composition of three isomorphisms. In this article, we establish the third isomorphism that relates the Seiberg-Witten Floer homology on the auxilliary manifold with the appropriate version of Seiberg-Witten Floer homology on the original manifold. This constitutes Theorem 4.1 in [KLT1], re-stated in a more refined form as Theorem 1.1 below. The tool used in the proof is a filtered variant of connected sum formula for Seiberg-Witten Floer homology, in special cases where one of the summand manifolds is $S^1 \times S^2$. Nevertheless, the arguments leading to the afore-mentioned connected sum formula are general enough to establish a connected sum formula in the wider context of Seiberg-Witten Floer homology with non-balanced perturbations. This is stated as Proposition 6.2 below. Although what is asserted in this proposition has been known to experts for some time, a detailed proof has not appeared in the literature, and therefore of some independent interest.

53C07; 52C15

1 Introduction

To summarize what was done in the predecessors to this article, [KLT1]-[KLT4] cited above: the first article in this series outlined a program for a proof of Theorem 1.4, based on a concatenation of three isomorphisms. The first isomorphism (Theorem 2.3 in [KLT1]) relates a version of embedded contact homology on an an auxillary manifold to the Heegaard Floer homology on the original, and was accomplished in [KLT2]-[KLT3]. The second isomorphism (Theorem 3.4 in [KLT1]) relates the relevant version of the embedded contact homology on the auxilliary manifold and a version of the Seiberg-Witten Floer homology on this same manifold. This was established

in [KLT4]. This last installment of the $HM = HF$ series contains the proof of the third isomorphism, stated as Theorem 4.1 in [KLT1]. Part of the content of this paper are drawn from unpublished details of the proof of the second author's Corollary 8.4 in [L], which describes the behavior of certain Seiberg-Witten Floer homology under handle addition.

1.1 The main theorem and an outline of proof

Let M be a closed, connected, and oriented 3-manifold. Given a Spin^c structure $\mathfrak{s}$ on M, P. B. Kronheimer and T. S. Mrowka defined in [KM] three flavors of Seiberg-Witten Floer homology, $\widehat{HM}_*$, $\overline{HM}_*$, and $\widecheck{HM}_*$, modelling on three different versions of S^1-equivariant homologies. These homology groups have the structure of modules over the graded ring

$$\mathbf{A}_\dagger(M) := \mathbb{Z}[U] \otimes \bigwedge^*(H_1(M;\mathbb{Z})/\operatorname{Tors}),$$

where U has degree -2 and elements in $H_1(M;\mathbb{Z})/\operatorname{Tors}$ has degree -1. These modules are graded by an affine space over $\mathbb{Z}/c_{\mathfrak{s}}\mathbb{Z}$, where $c_{\mathfrak{s}} \in 2\mathbb{Z}^{\geq 0}$ is the divisibility of $c_1(\mathfrak{s})$, the first Chern class of the Spin^c structure $\mathfrak{s}$. Moreover, as $\mathbf{A}_\dagger(M)$-modules, these three flavor of Seiberg-Witten Floer homologies fit into a long exact sequence modelling on the fundamental exact sequence of S^1-equivariant Floer homologies. (Cf. Equation (3.4) in [KM]).

$$\cdots \widehat{HM} \to \overline{HM} \to \widecheck{HM} \to \cdots \tag{1.1}$$

This is called *the first fundamental exact sequence of HM* in this article. In [L], the second author defined a fourth flavor of Seiberg-Witten Floer homology $\widetilde{HM}_*$ with the same module structure and relative grading. (It was originally denoted by HM^{tot} in [L]; given here as Definition 5.6). The definition models on the ordinary homology of an S^1-space. As such, it fits into a second long exact sequence together with $U\,\widehat{HM}_*$ and $\widehat{HM}_*$. This is referred to as the *second fundamental exact sequence of HM*; see Lemma 5.7 below.

In this article, we regard these four flavors of HM as a system, in the order of $\widehat{HM}_*$, $\overline{HM}_*$, and $\widecheck{HM}_*$, $\widetilde{HM}_*$. They are denoted collectively by $\mathring{HM}_*$.

As will be detailed in the upcoming Section 2, the Seiberg-Witten Floer homologies (also referred to as the monopole Floer homology in this article) $\mathring{HM}_*$ depends on the cohomology class of the perturbation form ϖ in addition to the Spin^c structure $\mathfrak{s}$. One may also define a monopole Floer homology with local coefficients Γ compatible with $\mathfrak{s}$ and $[\varpi]$. Of particular interest to us is the case when the perturbation is "balanced";

in this case Γ may be taken to be $\mathbb{Z}$. These are denoted by $\mathring{HM}_*(M, \mathfrak{s}, c_b)$; and this is the variant of monopole Floer homology to be equated with the Heegaard Floer homology HF_*°, in Theorem 1.4 below. This is, in a sense, the strongest possible statement of equivalence between HM and HF, as the monopole Floer homology $\overline{HM} \neq 0$ and $\widehat{HM} \neq \widetilde{HM}$ only in the balanced case. The equivalence between other versions of HM and HF may be deduced from this case through the use of local coefficients. It is also worth mentioning that a coarser version of Seiberg-Witten-Floer homology, $HM_\bullet$, defined by taking a completion of the Floer complex with respect to grading, frequently appears in [KM] and other literature. In this article we work exclusively with the original version, $\mathring{HM}_*$.

The upcoming Theorem 1.1 relates $\mathring{HM}_*(M, \mathfrak{s}, c_b)$ with two filtered variants of monopole Floer homology. The first was introduced in [L], originally denoted by HMT° therein. Here, the label $\circ$ stands, in specific order, for $-, \infty, +, \wedge$. The fact that they appear in the superscript (instead of the top) of the notation, and the order in which they appear, reflects the nature of their definition. The latter is done following the algebraic framework of Ozsvath-Szabo in [OS1]. The second of these two variants was introduced in [KLT4] (Cf. also Section 4 of [KLT1] for a brief summary). They are denoted by $\mathrm{H}_*^\circ(Y)$ in [KLT1], and by H_{SW}° in [KLT4]. The construction of both these filtered monopole Floer homologies is based on the same general framework, which we describe in Section 3 below. This framework always produces four flavors of Floer homologies, labeled by $\circ = -, \infty, +, \wedge$; and they are related by two fundamental long exact sequences. (To be more precise, only the first three flavors appearred in [KLT1] and [KLT4], but it shall become clear in Section 3 that the afore-mentioned general construction actually give rise to a fourth flavor). The basic ingredient of this construction consists of a triple of data: A certain Spin^c 3-manifold Y_Z, a closed 2-form w on Y_Z used to define a monotone perturbation to the Seiberg-Witten equations, and a special 1-cycle γ embedded in Y_Z useful for defining a filtration on the associated monopole Floer complex. Further constraints on the choice of this triple are given in Section 3.2.

The triple that enters the definition of HMT° is what was denoted by $(\underline{M}, *d\underline{f}, \underline{\gamma})$ in [L]. Here, $\underline{M}$ is constructed from M by adding a 1-handle along the extrema of f, the latter being a Morse function giving rise to the Heegaard diagram used to define HF°. What was denoted by $\underline{f}$ is an S^1-valued harmonic Morse function obtained by a natural extension of f. The 1-circle $\underline{\gamma}$ is related to the path $\gamma_z \subset M$ used by Ozsvath-Szabo to define a filtration on the Heegaard Floer complex. The triple used for the definition of H_{SW}° in [KLT4] was denoted by $(Y, w, \gamma^{(z_0)})$ in [KLT4] and [KLT2]. The 3-manifold Y is obtained from $\underline{M}$ by attaching additional 1-handles along pairs of index 1 and index

2 critical points of f; the two-form w is constructed from a natural extension of $*df$. The 1-cycle $\underline{\gamma}$ in $\underline{M}$ becomes the 1-cycle $\gamma^{(z_0)}$ in Y after the handle-attachment. The precise definitions of HMT° and H°_{SW} may be found in Section 3.8. It turns out that both Floer homologies are equipped with $\mathbf{A}_\dagger(M)$-module structures, as are $HF^\circ(M)$ and $\mathring{HM}(M, \mathfrak{s}, c_b)$.

Let G denote the number of 1-handles added to $\underline{M}$ in order to obtain Y.

The main theorem of this article relates the three versions of monopole Floer homologies: $\mathring{HM}(M, \mathfrak{s}, c_b)$, HMT°, and H°_{SW}.

Theorem 1.1 *View $H_*(S^1) \simeq \bigwedge^* H_1(S^1)$ as a graded algebra as prescribed in Section 1.3 below. Then:*

(1) *There exists a system of isomorphisms of $\mathbf{A}_\dagger(M)$-modules*

$$\mathrm{H}^\circ(Y) \xrightarrow{\simeq} \mathrm{HMT}^\circ \otimes H_*(S^1)^{\otimes \mathrm{G}}, \quad \circ = -, \infty, +, \wedge,$$

that preserves the relative gradings and natural with respect to the fundamental long exact sequences on both sides.

(2) *There exists a system of canonical isomorphisms of $\mathbf{A}_\dagger(M)$-modules from*

$$\mathrm{HMT}^\circ, \ \circ = -, \infty, +, \wedge \text{ respectively to } \mathring{HM}(M, \mathfrak{s}, c_b), \ \circ = \wedge, -, \vee, \sim,$$

that preserves the relative gradings and natural with respect to the fundamental long exact sequences on both sides.

The proof of this theorem is given in Section 6.3. The remainder of this section gives a brief outline of this proof.

Given how Y is constructed from $\underline{M}$, and $\underline{M}$ in turn from M, it is little surprise that the preceding theorem is a consequence of certain filtered variant of connected sum formula for Seiberg-Witten-Floer homologies. See Propositions 6.4 in Section 6.3. The first steps of the proof of this formula, via understanding the chain maps on Seiberg-Witten Floer complexes induced by cobordisms associated to the connected sum, lead to a connected sum formula for Seiberg-Witten Floer homologies *sans filtration*. This is stated as Proposition 6.2 below.

The more essential part of the proof, which also constitutes the major technical component of this article, consists of an extension of the framework defining HMT° and $\mathrm{H}^\circ(Y)$ to the context of cobordisms and their assciated chain maps. The analytical foundation of such an extension is provided in Sections 7-9 of this article.

The proof of part (2) of Theorem 1.1 also involves some homological algebra computation that turns out to be a manifestation of so-called "Koszul duality". An elementary account of the relevant part of this story is given in Section 4. This algebraic machinery expresses all four flavors of the balanced monopole Floer homology, $\mathring{HM}(M, \mathfrak{s}, c_b)$ in terms of a balanced monopole Floer complex of the first flavor, $\widehat{CM}_*(M, \mathfrak{s}, c_b)$. Meanwhile, the filtered connected sum formula previously mentioned expresses all four flavors of HMT° in terms of a monopole Floer complex with "negative monotone" perturbation, $CM_*(M, \mathfrak{s}, c_-)$. See Proposition 5.9 below. These two monopole Floer complexes are linked via a chain-level variant of the following result of Kronheimer-Mrowka's:

Theorem 1.2 ([KM] Theorem 31.5.1) $\widehat{HM}_*(M, \mathfrak{s}, c_b) \simeq HM_*(M, \mathfrak{s}, c_-)$.

The right hand side of the preceding isomorphism refers to the monopole Floer homology for negative monotone perturbations. A brief account of this variant of monopole Floer homology can be found in Section 2.3. The construction of both H°_{SW} and HMT° are based on negative monotone monopole Floer complexes.

More on the motivation for various constructions in the article may be found in [L].

Remark 1.3 With the hind-sight gained from Juhasz's [Ju] and Kronheimer-Mrowka's [KM1] definitions of sutured Floer homologies, we feel that HMT° are best interpreted as variants of sutured Floer homology. In particular, $HM(M(1), \mathfrak{s}(1)) = \widehat{\mathrm{HMT}}(\underline{M}, \underline{\mathfrak{s}})$ in terms of the notation in [Ju], [KM1] and [L]. From this point of view, Theorem 1.1 (2) may be viewed as a re-interpretation of monopole Floer homology of closed 3-manifolds as (generalized) sutured Floer homology. In particular, the $\circ = \wedge$ variant of this statement is a Seiberg-Witten analog of Prop 2.2 in [Ju], where the hat-version of the Heegaard Floer homology is re-interpreted as a sutured Floer homology. See also Theorem 1.6 announced in [CGHH] for an *ECH* analog (of the $\circ = \wedge$ variant). We hope to discuss this in more detail elsewhere. (See also the end of Part 4 in Section 9.1).

1.2 Relating Heegaard and Seiberg-Witten Floer homologies

With all said and done, the main result of this articles combines with those in [KLT1]-[KLT4] to reach our ultimate goal:

Theorem 1.4 *Let M be a closed, oriented 3-manifold, and $\mathfrak{s}$ be a* Spin^c *structure on M. Then there exists a system of isomorphisms from $HF^\circ_*(M, \mathfrak{s})$, $\circ = -, \infty, +, \wedge$,*

respectively to $\mathring{HM}_*(M, \mathfrak{s}, c_b)$, $\circ = \wedge, -, \vee, \sim$, *as* $\mathbb{Z}/c_{\mathfrak{s}}\mathbb{Z}$*-graded* $\mathbf{A}_\dagger(M)$*-modules, which is natural with respect to the fundamental exact sequences of the Heegaard and monopole Floer homologies.*

The result summaries the relation between the Heegaard and monopole Floer homlogies, which has been conjectured since the inception of Heegaard Floer theory. See for example Conjecture 1.1 in [OS2] , I.3.12 in [KM], Conjecture 1 in [KMan], and Conjecture 1.1 in [L].

As the Heegaard Floer homology HF° makes no other appearances for the rest of this article, the reader is referred to [OS1] and [OS2] for its definition and properties. In particular, the fundamental exact sequences relating its four flavors take the following form:

$$\cdots \to HF^- \to HF^\infty \to HF^+ \to \cdots$$
$$\cdots \to HF^- \xrightarrow{U} HF^- \to \widehat{HF} \to \cdots$$

Proof of Theorem 1.4. An outline of the proof is already given in [KLT1]. To summarize, By combining the two parts of Theorem 1.1, one has (Cf. Theorem 4.1 in [KLT1]):

$$\mathrm{H}^\circ(Y) \simeq \mathring{HM}(M, \mathfrak{s}, c_b) \otimes H_*(S^1)^{\otimes \mathrm{G}}.$$

Meanwhile, it is asserted in Theorem 3.4 of [KLT1] and proven in [KLT4] that the left hand side of the above isomorphism is in turn isomorphic to what was called "*ech*$^\circ$". Finally, the latter is claimed in Theorem 2.4 of [KLT1], and proven in [KLT2]-[KLT3], to be isomorphic to $HF^\circ(M\mathfrak{s}) \otimes H_*(S^1)^{\otimes \mathrm{G}}$. Each of the three isomorphisms above preserve the $\mathbf{A}_\dagger(M)$-module structure and the fundamental exact sequences on both sides. □

1.3 Some notations and conventions

Throughout the remainder of this paper, section numbers, equation numbers, and other references from [KLT1]-[KLT4] are distinguished from those in this paper by the use of the appropriate Roman numeral as a prefix. For example, 'Section II.1' refers to Section 1 in [KLT2]. In addition, the following conventions are used:

- As in [KLT1]-[KLT4], we use c_0 to denote a constant in $(1, \infty)$ whose value is independent of all relevant parameters. The value of c_0 can increase between subsequent appearances.

- As in [KLT1]-[KLT4], we denote by χ a fixed, non-increasing function on $\mathbb{R}$ that equals 1 on a neighborhood of $(-\infty, 0]$ and equals 0 on a neighborhood of $[1, \infty)$.

- When left unspecified, the modules, chain complexes and homologies in this article are over the coefficient ring $\mathbb{K}$, which can be taken to be $\mathbb{Z}$ as was done in [KLT1]-[KLT4]. Using a separate notion serves to distinguish different roles the abelian group $\mathbb{Z}$ in this article, e.g. as the group of deck transformations versus the coefficient ring of the chain complexes.

- The term "module" in this article refers to either a left module or a right module. Thus, both the monopole Floer homology and monopole Floer cohomology are said to have a module structure over the ring $H^*(BS^1)$. Note in contrast that in [KM], a "module" refers specifically to a left module. Moreover, what appears as $U_\dagger$ in [KM] is denoted by U in this article for simplicity, since we focus on Floer homology as opposed to cohomology.

- The definition of Floer complexes in this article often depends on several parameters, yet there are chain homotopies relating the Floer complexes with the values of some of the parameters changed. In the interest of simplicity, these parameters are usually left unspecified in our notation for the Floer complexes unless necessary.

- Due to geometric motivations (cf. [GKM]), we view $H_*(S^1)$ and $H^*(BS^1)$ both as free commutative differential graded algebras with zero differential and a single generator, where the odd generator y for $H_1(S^1)$ has degree 1, while the even generator u for $H^*(BS^1)$ has degree -2. In this section commutativity and the commutator $[\cdot, \cdot]$ are meant in the graded sense. In particular, what is called an "anti-chain map" in [KM] is in our terminology an odd chain map.

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2 Elements of Seiberg-Witten Floer theory

This subsection reviews some backgrounds on Seiberg-Witten-Floer theory, with the book [KM] as the definitive reference. By way of this, we introduce some notation and terminlogy used in the rest of this article, some of which differ from those in [KM]. We focus mostly on the special cases involved in the proof of Theorem 1.1, leaving the general details for the reader to consult [KM]. Many notions here have analogs in, e.g. [KLT4], [LT], which work with similar settings.

2.1 Seiberg-Witten equations on 3-manifolds

Let M be a closed, oriented, Riemannian 3-manifold. Fix a Spin^c-structure $\mathfrak{s}$ on M and let $\mathbb{S}$ denote its associated spinor bundle. We call a pair, $(\mathbb{A}, \Psi)$, consisting of a Hermitian connection on $\det(\mathbb{S})$ and a section of $\mathbb{S}$ a (Seiberg-Witten) *configuration*. The gauge group $C^\infty(M; U(1))$ acts on the space of configurations in the following fashion: Let $u : M \to U(1)$. Then u sends a configuration, $(\mathbb{A}, \Psi)$, to $(\mathbb{A} - 2u^{-1}du, u\Psi)$. Two solutions obtained one from the other in this manner are said to be *gauge equivalent*. Note that this $C^\infty(M; U(1))$ action is free except at pairs of the form $(\mathbb{A}, \Psi = 0)$; these are called *reducible* configurations. Configurations which are not reducible is *irreducible*.

In the most general form, the 3-dimensional Seiberg-Witten equations ask that a configuration $(\mathbb{A}, \Psi)$ obey

$$\begin{cases} B_{\mathbb{A}} - \Psi^\dagger \tau \Psi + i\varpi - \mathfrak{T} = 0 \quad \textit{and} \\ D^{\mathbb{A}}\Psi - \mathfrak{S} = 0, \end{cases} \tag{2.1}$$

where $B_{\mathbb{A}}$ denotes the Hodge dual of the curvature form of $\mathbb{A}$, $D^{\mathbb{A}}$ denotes the Dirac operator, and the quadratic term $\Psi^\dagger \tau \Psi$ is as in Section 1.2 of [LT]. ϖ is a closed 2-form, and the pair $(\mathfrak{T}, \mathfrak{S})$ is a small perturbation arising as the formal gradient of a gauge-invariant function of $(\mathbb{A}, \Psi)$. This is called a tame perturbation in [KM], and is in general needed to guarantee the transversality properties necessary for the definition of Seiberg-Witten-Floer homology. See Chapters 10 and 11 in [KM]. In the simplest case, $(\mathfrak{T}, \mathfrak{S})$ may be taken to be of the form

$$(\mathfrak{T}, \mathfrak{S}) = (2i * d\mu, 0) \tag{2.2}$$

for a smooth 1-form μ taken from a Banach space called Ω in [KLT4]. This may be assumed to be a subspace of the Banach space of tame perturbations in Chapter 11.6

in [KM], and hence inherits the so-called "$\mathcal{P}$-norm" from [KM]. This norm bounds the norms of the derivatives of μ to any given order.

Irreducible solutions to (2.1) may exist only when the cohomology class $[\varpi] = 2\pi c_1(\det \mathbb{S})$. In this case the Seiberg-Witten equations (2.1) is said to have *balanced perturbation*, while it is said to have *exact perturbation* when $[\varpi] = 0$. The cases when $[\varpi] = 2\mathrm{r} c_1(\det \mathbb{S})$ is said to be *monotone*: when $\mathrm{r} > \pi$ it is said to be *negative monotone*, and when $\mathrm{r} < \pi$ it is said to be *positive monotone*. Note that when $c_1(\det \mathbb{S})$ is torsion, the notation of balanced, exact, and positive or negative monotone perturbations are equivalent. We work in the negative monotone case with nontorsion $c_1(\det \mathbb{S})$ for most part of this paper where all Seiberg-Witten solutions are irreducible. Note in contrast that in the closely related series of articles [T1]-[T5], ϖ is taken to be da for a contact 1-form a, which is an exact perturbation.

This said, unless otherwise specified, from now on we set

$$\varpi = 2\mathrm{r}w \tag{2.3}$$

for a closed 2-form w in the cohomology class of $c_1(\det \mathbb{S})$ and a real number $\mathrm{r} > \pi$. When $c_1(\det \mathbb{S})$ is torsion, we always set $w \equiv 0$. Otherwise, the particulars of w for the proof of our main theorem 1.1 are described in Section 3.2.

To make contact with the notation in [KLT4], write

$$\det(\mathbb{S}) = E^2 \otimes K^{-1}$$

with $K \to M$ being a fixed complex line bundle. Fix a smooth connection, A_K, on K^{-1}. Where w is nonvanishing (such as over the stable Hamiltonian manifold Y in [KLT4]), K^{-1} is typically given by $\mathrm{Ker}\,(*w) \subset TM$ and E the $i|w|$-eigen-bundle of the Clifford action by w. More constraints on the choice of K and A_K will be specified along the way through the rest of this article.

With A_K chosen, let A denote the connection on the E-summand corresponding to $\mathbb{A}$, and write $\Psi = \sqrt{2\mathrm{r}}\,\psi$. In this case, perturbations of the form (2.2) suffice for our purpose. Since the Riemannian metric and a connection on E determine a Spin^c connection on $\mathbb{S}$, we often consider the equivalent equations for (A, ψ) of the form

$$\begin{cases} B_A - \mathrm{r}\,(\psi^\dagger \tau \psi - i * w) + \frac{1}{2} B_{A_K} - i * d\mu = 0, \\ D_A \psi = 0, \end{cases} \tag{2.4}$$

where $D_A = D^{\mathbb{A}}$, B_A is the Hodge star of the curvature 2-form of A and B_{A_K} denotes the Hodge star of the curvature 2-form for the connection, A_K.

Given a Hermitian line bundle $V \to M$, we use $\mathrm{Conn}\,(V)$ to denote the space of Hermitian connections on V. The equations in (2.1) are the variational equations of the functional $\mathfrak{a}$ of $(A,\psi) \in \mathrm{Conn}\,(E) \times C^\infty(M;\mathbb{S})$, given by

$$\mathfrak{a} = \frac{1}{2}\mathfrak{cs} - \mathrm{r}\mathrm{w} + \mathfrak{e}_\mu + \mathrm{r}\int_M \psi^\dagger D_A\psi, \tag{2.5}$$

where the notation is as follows: The functions $\mathfrak{cs}$ and w are defined using a chosen reference connection on E. Let A_E denote the latter. With A written as

$$A = A_E + \hat{a}_A,$$

then w and $\mathfrak{cs}$ are given by

$$\mathrm{w} = i\int_M \hat{a}_A \wedge w \quad \text{and} \quad \mathfrak{cs} = -\int_M \hat{a}_A \wedge d\,\hat{a}_A - 2\int_{Y_Z} \hat{a}_A \wedge \big(F_{A_E} + \frac{1}{2}F_{A_K}\big). \tag{2.6}$$

What is denoted by $\mathfrak{e}_\mu$ is the integral over M of $i\mu \wedge F_A$. The functionals $\mathfrak{a}$, w and $\mathfrak{cs}$ in general is not invariant under the $C^\infty(M;\mathbb{S})$ action on $\mathrm{Conn}\,(\det\,\mathbb{S}) \times C^\infty(M;\mathbb{S})$, however their differentials descend to the orbit space. These differentials are henceforth denoted by $d\mathfrak{a}$, $d(\mathfrak{cs})$, $\cdots$, etc.

To define the Seiberg-Witten Floer homology in general, [KM] takes a real blow-up of the space $\mathrm{Conn}\,(\det\,\mathbb{S}) \times C^\infty(M;\mathbb{S})$ along the set of reducibles (Cf. Chapter 6 of [KM]). This blown-up space is denoted as $\mathcal{C}^\sigma(M,\mathfrak{s})$ therein and has a free $C^\infty(M,U(1))$-action. The functional $\mathfrak{a}$ extends to $\mathcal{C}^\sigma$ and the resulting variational equation of are then used to define the Seiberg-Witten equations.

A solution $\mathfrak{c}$ to the Seiberg-Witten equations or its corresponding gauge equivalence class $[\mathfrak{c}] \in \mathcal{C}^\sigma(M,\mathfrak{s})/C^\infty(M,U(1))$ is said to be *non-degenerate* when certain differential operator $\mathfrak{L}_\mathfrak{c}$ has trivial kernel. The explicit form of this operator is given for irreducible solutions of (2.1) in (7.30) below. In general, this notion of nodegeneracy arises from the interpretation of $[\mathfrak{c}]$ as a zero of the 1-form $d\mathfrak{a}$ on $\mathcal{C}^\sigma(M,\mathfrak{s})/C^\infty(M,U(1))$. With the metric and ϖ fixed, a choice of $(\mathfrak{T},\mathfrak{S})$, (or in the case of (2.4)), of μ) such that all solutions to (2.1) or (2.4) are non-degenerate is said in what follows to be *suitable.* In the negative monotone case with nontorsion $c_1(\det\,\mathbb{S})$, a suitable choice for μ can be found with $\mathcal{P}$-norm bounded by any given positive number. (Cf. e.g. (1.18) in [KLT4] and references therein). Otherwise, especially when reducible solutions exist, a suitable pair $(\mathfrak{T},\mathfrak{S})$ is typically of more general form than that of (2.2). Nondegenerate gauge-equivalence classes of reducible Seiberg-Witten solutions are further classified into the "stable" and "unstable" types in [KM].

2.2 Seiberg-Witten equations on 4-dimensional cobordisms

Let Y_-, Y_+ be closed oriented 3-manifolds. In this paper X will denote a simple cobordism from Y_- to Y_+ of the following sort: X is an oriented complete 4-manifold equipped with the extra structure listed below.

(2.7)

- There is a proper function $s\colon X \to \mathbb{R}$ with non-degenerate critical points with at most one single critical value, 0.
- There exists an orientation preserving diffeomorphism between the $s < 0$ part of X and $(-\infty, 0) \times Y_-$ that identifies s with the Eulidean coordinate on the $(-\infty, 0)$ factor.
- There exists an orientation preserving diffeomorphism between the $s > 0$ part of X and $(0, \infty) \times Y_+$ that identifies s with the Eulidean coordinate on the $(0, \infty)$ factor.
- There is an even class in $H^2(X; \mathbb{Z})$ that restricts to the $s < 0$ and $s > 0$ parts of X as the respective Y_- and Y_+ versions of $c_1(\det(\mathbb{S}))$.

The diffeomorphism in the second bullet of (2.7) is used, often implicitly, to identify the $s < 0$ part of X with $(-\infty, 0) \times Y_-$; and the diffeomorphism in the third bullet of (2.7) is likewise used to identify the $s > 0$ part with $(0, \infty) \times Y_+$. Fix a class satisfying the last bullet of (2.7) and denote it also by $c_1(\det(\mathbb{S}))$.

Assume that the Riemannian metric on X satisfies the following:

(2.8)

- There exists $L \geq 100$ such that the metric on the $s \leq -L$ and $s \geq L$ parts of X are identified by the embeddings in the second and third bullets of (2.7) with the respective product metrics on $(-\infty, -L] \times Y_-$ and $[L, \infty) \times Y_+$.
- The metric pulls back from the $|s| \in [L - 8, L]$ part of X via the embeddings from the second and third bullets of (2.7) as the quadratic form $ds^2 + \mathfrak{g}$ with $\mathfrak{g}$ being an s-dependent metric on either Y_- or Y_+ as the case may be.

The chosen metric on X is used to write $\bigwedge^2 T^*X$ as $\Lambda^+ \oplus \Lambda^-$ with Λ^+ denoting the bundle of self-dual 2-forms and with Λ^- denoting the corresponding bundle of anti-self dual 2-forms. A given 2-form $\mathfrak{w}$ is written with respect to this splitting as $\mathfrak{w} = \mathfrak{w}^+ + \mathfrak{w}^-$.

Use the metric to define the notion of a Spin^c-structure on X. It follows from the last bullet in (2.7) that there is a Spin^c structure that restricts to the $s \leq -2$ and $s \geq 2$ parts

of X as the given Spin^c structures from Y_- and Y_+, and has its first Chern class equal to $c_1(\det(\mathbb{S}))$. Fix such a Spin^c structure and use $\mathbb{S}^+$ and $\mathbb{S}^-$ to denote the respective bundles of self-dual and anti-self dual spinors.

The Seiberg-Witten equations on X are equations for a pair $(\mathbb{A},\Psi)$ with $\mathbb{A}$ being a Hermitian connection on the line bundle $\det(\mathbb{S}^+)$ and with Ψ being a section of $\mathbb{S}^+$. The relevant version reads

$$F_{\mathbb{A}}^+ - (\Psi^\dagger\tau\Psi - i\varpi_X) - \mathfrak{T}^+ = 0 \quad \text{and} \quad \mathcal{D}_{\mathbb{A}}^+\Psi - \mathfrak{S}^+ = 0, \tag{2.9}$$

where the notation uses $F_{\mathbb{A}}$ to denote the curvature 2-form of $\mathbb{A}$, and it uses $\Psi^\dagger\tau\Psi$ to denote the bilinear map from $\mathbb{S}^+$ to $i\Lambda^+$ that is defined using the Clifford multiplication. Meanwhile, $\mathcal{D}_{\mathbb{A}}^+\colon \Gamma(\mathbb{S}^+)\to\Gamma(\mathbb{S}^-)$ and $\mathcal{D}_{\mathbb{A}}^-\colon \Gamma(\mathbb{S}^-)\to\Gamma(\mathbb{S}^+)$ are the 4-dimensional Dirac operators on X defined by the metric and the chosen connection $\mathbb{A}$. What is denoted by ϖ_X is a self-dual 2-form satisfying the following list for some $L'\geq L$:

(2.10)
- The pull-back of ϖ_X from the $s<-L'$ part of X via the embedding from the second bullet of (2.7) is twice the self dual part of a closed 2-form ϖ_- on Y_-.
- The pull-back of ϖ_X from the $s>L'$ part of X via the embedding from the third bullet of (2.7) is twice the self dual part of a closed 2-form ϖ_+ on Y_+.

The pair $(\mathfrak{T}^+,\mathfrak{S}^+)$ is the 4-dimensional analog of $(\mathfrak{T},\mathfrak{S})$ in (2.1); see (24.2) in [KM].

An important special case is when (2.9) is defined on a *product cobordism*. By this we mean that $X=\mathbb{R}\times M$ for a closed oriented Spin^c 3-manifold M, with the function s as the Euclidean coordinate of the $\mathbb{R}$ factor; the Riemannian metric on X is the product of the affine metric on $\mathbb{R}$ and the Riemannian metric on M, and both ϖ_X and $(\mathfrak{T}^+,\mathfrak{S}^+)$ are invariant under the natural $\mathbb{R}$-action on $\mathbb{R}\times M$. Thus, the conditions in the first bullet of (2.8) and in (2.10) may be paraphrased as saying that the $s^{-1}[L',\infty)$ and $s^{-1}(-\infty,-L']$ part of the Seiberg-Witten equations on X are those of product cobordisms. As explained in [KM], Clifford action by ds over product cobordisms may be used to identify $\mathbb{S}^+\simeq\mathbb{S}^-$. Meanwhile, both are the pull-back of a spinor bundle $\mathbb{S}$ over M. In this way, (2.9) may be re-written as a gradient flow equation of the action functional $\mathfrak{a}$, cf. (IV.1.20).

A solution $\mathfrak{d}=(\mathbb{A},\Psi)$ to (2.9) is said to be an *instanton* if the constant $s\leq -L$ pull-backs converge as $s\to-\infty$ to a pair that can be written as $(\mathbb{A}_-,\Psi_-)$, with $(\mathbb{A}_-,\Psi_-)$ being a solution to (2.1) on Y_-; and if the constant $s\geq L$ pull-backs converge as $s\to\infty$ to a pair $(\mathbb{A}_+,\Psi_+)$, with $(\mathbb{A}_+,\Psi_+)$ being a solution to (2.1) on Y_+. If $\mathfrak{d}$ is an instanton then the convention in what follows will be to say that the respective $s\to-\infty$

and $s \to \infty$ limits of $\mathfrak{d}$ are $(\mathbb{A}_-, \Psi_-)$ and $(\mathbb{A}_+, \Psi_+)$. As in the 3-dimensional case, [KM] define a real "blow-up" of the space Conn $(\det \mathbb{S}^+) \times C^\infty(X, \mathbb{S}^+)$, this denoted by $\mathcal{C}^\sigma(X)$. The 4-dimensional Seiberg-Witten equations (2.9) may be generalized to elements in $\mathcal{C}^\sigma(X)$, and hence also the notion of an instanton. An (generalized) instanton has its $s \to -\infty$ and $s \to \infty$ limits in $\mathcal{C}^\sigma(Y_-)$ and $\mathcal{C}^\sigma(Y_+)$ respectively, in the sense explained in p.29 in [KM]. An instanton $(\mathbb{A}, \Psi)$ is said to be *reducible* when $\Psi \equiv 0$; otherwise it is *irreducible*.

The perturbation $(\mathfrak{T}^+, \mathfrak{S}^+)$ is introduced in (2.9) so that a certain operator that is associated to any given instanton solutions to (2.9) is Fredholm with trivial cokernel. Cf. Chapter 24.3 of [KM] in general and Equation (1.21) in [KLT4] for a special case closely related to this article. Instanton solutions with this property are said to be non-degenerate. We call perturbation term *suitable* when all instanton solutions to the corresponding version of (2.9) are non-degenerate. A suitable perturbation can be found for (2.9) with norm bounded by any given positive number. The relevant norm is also called the $\mathcal{P}$-norm. As in the case with elements in Ω, the $\mathcal{P}$-norm of a perturbation term bounds the norms of its derivatives to all orders.

Just as in the 3-dimensional case, in the case of 4-dimensional cobordisms we consider ϖ_X and pairs $(\mathfrak{T}^+, \mathfrak{S}^+)$ of the following form:

$$\varpi_X = 2rw_X \quad \text{and} \quad (\mathfrak{T}^+, \mathfrak{S}^+) = (i\mathfrak{w}_\mu^+, 0).$$

Here, $\mathfrak{w}_\mu$ is a 2-form of the form $d\big(\chi(L+s)\mu_- + \chi(L-s)\mu_+\big)$ for some 1-forms μ_-, μ_+ on Y_-, Y_+ respectively. However, in the case of a product cobordism $X = \mathbb{R} \times M$, we take $\mathfrak{w}_\mu = d\mu_- = d\mu_+$. Meanwhile, w_X is a self-dual 2-form constrained by the properties listed in (2.11) below, among others. These constraints involve another constant denoted by L_{tor} below. The latter is no smaller than $L+4$. The constraints use X_{tor} to denote the union of the components of the $|s| > 0$ part of X where $c_1(\det \mathbb{S})$ is torsion.

(2.11)

- The pull-back of w_X to each constant s slice of X is a closed 2-form whose de Rham cohomology class is that of $c_1(\det(\mathbb{S}))$.
- The embedding from the first bullet of (2.10) pulls back w_X from the $s < -L$ part of $X - X_{tor}$ as twice the self dual part of the Y_- version of the 2-form w. The embedding from the second bullet of (2.10) pulls back w_X from the $s > L$ part of $X - X_{tor}$ as twice the self dual part of the Y_- version of the 2-form w. The 2-form w_X is identically zero on any component of the $|s| > L_{tor}$ part of X_{tor}.

Similarly to the 3-dimensional case, the 4-dimensional Seiberg-Witten equations may be rewritten in terms of the pair $(A, \psi) \in \text{Conn}(E) \times C^\infty(\mathbb{S}^+)$ that is obtained from

the pair $(\mathbb{A}, \Psi) \in \mathrm{Conn}\,(E) \times C^\infty(\mathbb{S}^+)$ via the same formulae as those in the previous subsection. This requires an extension of K and A_K from the ends $s^{-1}[L', \infty) \cup s^{-1}(-\infty, -L']$. Constraints on such choices will be introduced in subsequent sections as needs arise; typically where ϖ_X is non-vanishing, E is chosen to be the $i|\varpi_X|$-eigenbundle under the Clifford action of ϖ_X on $\mathbb{S}^+$.

2.3 The monopole Floer chain complex

Fix a closed, oriented, connected Riemannian 3-manifold M and a Spin^c structure $\mathfrak{s}$ on it. Suppose for now that $\mathfrak{s}$ has non-torsion first Chern class, and ϖ, $(\mathfrak{T}, \mathfrak{S})$ are as in (2.3) and (2.2) respectively, with $\mathrm{r} > \pi$. Fix also a complex Hermitian line bundle $K \to M$ as specified in Section 2.1 above. The *spectral flow* function on $\mathrm{Conn}\,(E) \times C^\infty(M; \mathbb{S})$ is defined initially on the complement of a certain codimension 1 subvariety just as in Section 1e in [KLT4] using a chosen Hermitian connection on E and a suitably generic section of $\mathbb{S}$. As such it is locally constant and integer-valued. The definition can be extended to the whole of $\mathrm{Conn}\,(E) \times C^\infty(M; \mathbb{S})$ as explained in Sections 7.6 and 7.8 below. This spectral flow function is denoted by $\mathfrak{f}_s$. It suffices for now to know only that this extended function $\mathfrak{f}_s$ has integer values and that the functions

$$\mathfrak{cs}^{\mathfrak{f}} := \mathfrak{cs} - 4\pi^2 \mathfrak{f}_s \quad \text{and} \quad \mathfrak{a}^{\mathfrak{f}} := \mathfrak{a} + 2\pi(\mathrm{r} - \pi)\mathfrak{f}_s$$

are invariant under the action of $C^\infty(M; \mathbb{S})$ on $\mathrm{Conn}\,(E) \times C^\infty(Y; \mathbb{S})$ that has a $\hat{u} \in C^\infty(M; \mathbb{S})$ sending (A, ψ) to $(A - \hat{u}^{-1} d\hat{u}, \hat{u}\psi)$. By way of comparison, $\mathfrak{a}$, $\mathfrak{f}_s$, and $\mathfrak{cs}$ are not invariant under this action.

Denote by $\mathcal{Z}_{w,\mathrm{r}}$ the set of gauge-equivalence classes of solutions to the corresponding (2.4). (This was denoted by a slightly different notation, $\mathcal{Z}_{SW,\mathrm{r}}$, in [KLT4]). It is well-known that in this case, for a generic choice of r, μ, this set $\mathcal{Z}_{w,\mathrm{r}}$ consists of finitely many, nondegenerate irreducible elements. (Cf. e.g. (IV.1.18) and references therein, ignoring the "holonomy nondegenerate" condition there for the moment). Assume this to be the case. Consider next the 4-dimensional Seiberg-Witten equations on the product cobordism $\mathbb{R} \times M$, with $w_X = 2w^+$, and $\mu_- = \mu_+ = \mu$. Here, w is used to denote the pull-back of the 2-form w on M under the projection of $\mathbb{R} \times M$ to its second factor. Given an instanton $\mathfrak{d}$ on this product cobordism with $s \to -\infty$ and $s \to \infty$ limits given respectively by representatives of $\mathfrak{c}_-$, $\mathfrak{c}_+$ in $\mathcal{Z}_{w,\mathrm{r}}$. The differential operator in (IV.1.21) has a Fredholm extension, whose index we denote by $\imath_{\mathfrak{d}}$. By [APS], in this case

$$\imath_{\mathfrak{d}} = \mathfrak{f}_s(\mathfrak{c}_+) - \mathfrak{f}_s(\mathfrak{c}_-). \tag{2.12}$$

Let $\mathcal{M}_k(\mathfrak{c}_-, \mathfrak{c}_+)$ denote the space of such instantons with $\imath_{\mathfrak{d}} = k$. These spaces are k-dimensional manifolds with a free $\mathbb{R}$-action when the perturbation term in the Seiberg-Witten equations is suitable and $k > 0$. In particular, the monotonicity assumption guarantees that $\mathcal{M}_1(\mathfrak{c}_-, \mathfrak{c}_+)/\mathbb{R}$ consists of finitely many elements. With a coherent orientation chosen (this amounts to choices of preferred elements of $\Lambda(\mathfrak{c})$ for all $\mathfrak{c} \in \mathcal{Z}_{w,\mathrm{r}}$ in the language of [KM]), each element in $\mathcal{M}_1(\mathfrak{c}_-, \mathfrak{c}_+)/\mathbb{R}$ is assigned a sign.

Fix a ring $\mathbb{K}$, which can be taken to be $\mathbb{Z}$ for the rest of this article. The chain module for the monopole (or alternatively, Seiberg-Witten) Floer chain group is the free $\mathbb{K}$-module generated by $\mathcal{Z}_{w,\mathrm{r}}$, denoted by $\mathbb{K}(\mathcal{Z}_{w,\mathrm{r}})$ below. The spectral flow function $\mathfrak{f}_s$ descends to define a relative $\mathbb{Z}/c_{\mathfrak{s}}\mathbb{Z}$-grading on this module, where $c_{\mathfrak{s}} \in 2\mathbb{Z}$ is the divisibility of the first Chern class of the Spin^c structure $\mathfrak{s}$. The differential $\partial_{w,\mathrm{r}}$ of ths monopole Floer complex in this situation is the endomorphism of $\mathbb{K}(\mathcal{Z}_{w,\mathrm{r}})$ given by the rule

$$\mathfrak{c}_1 \mapsto \sum_{\mathfrak{c}_2 \in \mathcal{Z}_{w,\mathrm{r}}} \sum_{\mathfrak{d} \in \mathcal{M}_1(\mathfrak{c}_1, \mathfrak{c}_2)/\mathbb{R}} \mathrm{sign}(\mathfrak{d})\, \mathfrak{c}_2. \tag{2.13}$$

The afore-mentioned properties of $\mathcal{Z}_{w,\mathrm{r}}$ and $\mathcal{M}_1(\mathfrak{c}_1, \mathfrak{c}_2)/\mathbb{R}$ for suitable monotone perturbations guarantee that this homomorphism is well-defined, and it is of degree 1 according to (2.12). A typical gluing argument shows that $\partial_{w,\mathrm{r}}^2 = 0$. The homology of the above monopole Floer complex is the monopole Floer homology, or alternatively, the Seiberg-Witten-Floer homology. This is denoted as $HM_*(M, \mathfrak{s}, c_-)$ below.

As final remarks to this subsection, note that in [KM] there is an equivalent, geometric version of grading for the monopole Floer complexes in terms of homotopy classes of oriented 2-plane fields. This is briefly described in Part 1 of Section 6.1 below, and denoted by $\mathbb{J}(M)$ therein. A very brief description of this in the special cases relevant to this article will appear in Part 1 of Section 6.1. Meanwhile, the signs $\mathrm{sign}(\mathfrak{d})$ assigned to the rules in [KM] depend on a choice of *homology orientation* of M. See Definition 22.5.2 in [KM].

2.4 $\mathbf{A}_\dagger$-module structure of monopole Floer complexes

As already mentioned in Section 1, there is an $\mathbf{A}_\dagger$-module structure on the monopole Floer homology. This subsection describes the underlying chain maps.

Part 1: The generators of $\mathbf{A}_\dagger$ as an algebra, the degree -2 element U and elements of a basis $\{\mathfrak{t}_i\}_i$ for $H_1(M;\mathbb{Z})/\,\mathrm{Tors}$, are induced from chain maps from the underlying

monopole Floer complex to itself, which we denote by the same notation, U and $\mathfrak{t}_i$. These are all defined by considering higher dimensional $\mathcal{M}_k$. Nevertheless, there are many possible ways of defining the chain map U or chain maps $\mathfrak{t}_i$. It turns out that they are all chain-homotopic, and we use the same notation for any one of these possible definitions indescriminantly unless absolutely necessary. In the monotone, non-balanced cases, certain possible definitions of these chain maps are given in Part 7 of Section IV.1.c. The definitions therein depend on the choice of a point $p \in M$ for the U-map, and choices of embedded circles $\gamma_i \subset M$ with homology class mod torsion $[\gamma_i] = \mathfrak{t}_i$ for the $\mathfrak{t}_i$ maps.

Part 2: For the convenience of later discussion, it is useful to note that the p-dependent U-map given in Part 7 of Section IV.1.c has the following alternative description: Due to the free $\mathbb{R}$-action on $\mathcal{M}_2(\mathfrak{c}_1, \mathfrak{c}_2)$ and the compactness properties of Seiberg-Witten moduli spaces, the latter may be written as $\mathbb{R} \times \mathcal{N}_1(\mathfrak{c}_1, \mathfrak{c}_2)$ for a compact 1-manifold $\mathcal{N}_1(\mathfrak{c}_1, \mathfrak{c}_2)$. Trivialize $E|_{\mathbb{R}\times M}$ by identifying it with the pull-back of the bundle $E \to M$ via the projection $\mathbb{R} \times M \to M$, and let $\mathrm{hol}_{\hat{p}}(A, \psi)$ denote the holonomy of A along $\mathbb{R} \times \{p\} \subset \mathbb{R} \times M$ with respect to this trivialization. Then $\mathrm{hol}_{\hat{p}}$ defines a map from $\mathcal{N}_1(\mathfrak{c}_1, \mathfrak{c}_2) = \mathcal{M}_2(\mathfrak{c}_1, \mathfrak{c}_2)/\mathbb{R}$ to S^1. Let $\mathrm{u}(\mathfrak{c}_1, \mathfrak{c}_2)$ denote the degree of this map. Then $\langle \mathfrak{c}_2, U\mathfrak{c}_1\rangle = \mathrm{u}(\mathfrak{c}_1, \mathfrak{c}_2)$. This definition of U-map applies both in the non-balanced case and balanced case.

To relate this definition with the one given in Section IV.1.c in the non-balanced case, recall that in that article $\mathcal{M}_{2,p}(\mathfrak{c}_1, \mathfrak{c}_2) \subset \mathcal{M}_2(\mathfrak{c}_1, \mathfrak{c}_2)$ denotes the subset of instantons $\mathfrak{d} = (A, (\alpha, \beta))$ with $\alpha(0, p) = 0$. In Section IV.1.c, $\langle \mathfrak{c}_2, U\mathfrak{c}_1\rangle$ is taken to be the signed count of elements in $\mathcal{M}_{2,p}(\mathfrak{c}_1, \mathfrak{c}_2)$. The latter is the zero set of the map $\hat{h}\colon \mathcal{M}_2(\mathfrak{c}_1, \mathfrak{c}_2) \to E|_{(0,p)} \simeq \mathbb{C}$ given by taking the value of α at the point $(0, p) \in \mathbb{R} \times M$. Write $\mathfrak{c}_1 = (A_-, (\alpha_-, \beta_-))$ and $\mathfrak{c}_2 = (A_+, (\alpha_+, \beta_+))$. The monotone, non-balanced assumption together with the constraint $\imath_{\mathfrak{d}} = 2$ implies that neither α_- nor α_+ may be identically zero. One may then choose p so that both $\alpha_-(p) \neq 0$ and $\alpha_+(p) \neq 0$. Due to the uniform L^∞-bound on Seiberg-Witten solutions, the afore-mentioned map $\hat{h}$ takes value in a compact region in $\mathbb{C}$, and it sends one end of $\mathcal{M}_2(\mathfrak{c}_1, \mathfrak{c}_2) \simeq \mathbb{R} \times \mathcal{N}_1(\mathfrak{c}_1, \mathfrak{c}_2)$, that of $\{-\infty\} \times \mathcal{N}_1(\mathfrak{c}_1, \mathfrak{c}_2)$, to the same point $\alpha_-(p) \in \mathbb{C}$, while sending the other end, $\{\infty\} \times \mathcal{N}_1(\mathfrak{c}_1, \mathfrak{c}_2)$, to a circle centered at 0 passing through $\alpha_+(p)$. Denote this circle by S_+. Thus, $\hat{h}$ extends to define a map from

$$\check{\mathcal{M}} := \big(\{-\infty\} \times \mathcal{N}_1(\mathfrak{c}_1, \mathfrak{c}_2) \cup \mathcal{M}_2(\mathfrak{c}_1, \mathfrak{c}_2) \cup \{\infty\} \times \mathcal{N}_1(\mathfrak{c}_1, \mathfrak{c}_2)\big)/\big((-\infty, \mathfrak{d}) \sim (-\infty, \mathfrak{d}')\big)$$

to $\mathbb{C}$, denoted by the same notation $\hat{h}$. Note that $\check{\mathcal{M}}$ is homeomorphic to finite copies of (oriented) 2-disks, and $\hat{h}$ sends $\partial\check{\mathcal{M}}$ to $S_+ \subset \mathbb{C} - \{0\}$. Consequently,

$\hat{h}^{-1}(0) = \mathcal{M}_{2,p}(\mathfrak{c}_1, \mathfrak{c}_2)$ has a well-defined Euler characteristic. The latter number is the degree of the map $\hat{h}|_{\partial\check{\mathcal{M}}}$, which is precisely the degree of the holonomy map $\mathrm{hol}_{\hat{p}}$, namely $\mathrm{u}(\mathfrak{c}_1, \mathfrak{c}_2)$.

2.5 Monopole Floer complexes for balanced perturbations

What follows next is a brief review of the more general situation, that applies to the balanced case. For details see Chapters VI and VIII in [KM]. As already mentioned in Section 2.1, [KM] considered the extension of (2.4) to $\mathcal{C}^\sigma$. The set of gauge equivalence classes of solutions to this extended Seiberg-Witten equation is denoted by $\mathfrak{C}$. Suppose that the perturbation to the Seiberg-Witten equation is suitable. The subsets of irreducible, unstable reducible, and stable reducible elements are respectively denoted by $\mathfrak{C}^o$, $\mathfrak{C}^u$, $\mathfrak{C}^s$. (In the nonbalanced situation previously considered, $\mathfrak{C} = \mathfrak{C}^o = \mathcal{Z}_{\mathrm{r},w}$). The first three flavors of monopole Floer homology as defined in [KM] use different combinations of $\mathfrak{C}^o$, $\mathfrak{C}^u$, $\mathfrak{C}^s$ to generate the chain groups: Set

$$C^o = \mathbb{K}(\mathfrak{C}^o), \quad C^u = \mathbb{K}(\mathfrak{C}^u), \quad C^s = \mathbb{K}(\mathfrak{C}^s), \quad \text{and let}$$
$$\hat{C} = C^o \oplus C^u, \quad \overline{C} = C^s \oplus C^u, \quad \check{C} = C^o \oplus C^s.$$

Meanwhile, the operator in (IV.I.21) has a Fredholm generalization for paths $\mathfrak{d}(s)$ in $\mathrm{Conn}\,(E) \times C^\infty(M, \mathbb{S})$ with $s \to \infty$ or $s \to -\infty$ limits that are nondegenerate elements in $\mathcal{C}^\sigma$. (See Sections 14.4 and 22.3 in [KM]). The index of this operator is also denoted by $\imath_{\mathfrak{d}}$ below; and it may be used to generalize the spectral flow function $\mathfrak{f}_s$ to the set of nondegenerate elements in $\mathcal{C}^\sigma$. This in turn defines a relative $\mathbb{Z}/c_{\mathfrak{s}}$-grading, gr, on the modules C^o, C^u, C^s. The chain modules $\hat{C}$, $\overline{C}$, $\check{C}$ are also $\mathbb{Z}/c_{\mathfrak{s}}$-graded according to the following rule:

$$\hat{C} = \bigoplus_j \hat{C}_j, \quad \hat{C} = \bigoplus_j \hat{C}_j, \quad \hat{C} = \bigoplus_j \hat{C}_j, \quad \text{where}$$
$$\hat{C}_j = C^o_j \oplus C^u_j, \quad \overline{C}_j = C^s_j \oplus C^u_{j+1}, \quad \check{C}_j = C^o_j \oplus C^s_j.$$

Note that the $\overline{C}$ chain module above is graded by a modified grading $\overline{\mathrm{gr}}$, related to gr via Equation (22.15) in [KM]. To define the differentials, define homomorphisms $\partial^\sharp_\natural\colon C^\sharp \to C^\natural$ via rules similar to (2.13) by counting irreducible instantons with $\imath_{\mathfrak{d}} = 1$ whose $s \to -\infty$ and $s \to \infty$ limits are in $\mathfrak{C}^\sharp$ and $\mathfrak{C}^\natural$ respectively; see [KM] Equation (22.8) for the precise formulae. Here, $\sharp$ and $\natural$ may stand for one of the labels u, o, s; however, due to the way $\mathfrak{C}^u$, $\mathfrak{C}^s$, $\mathfrak{C}^o$ are defined, only the homomorphisms ∂^o_o, ∂^o_s, ∂^u_o, ∂^u_s are nontrivial. Meanwhile, there are homomorphisms $\overline{\partial}^\sharp_\natural\colon C^\sharp \to C^\natural$, and with $\sharp$ and $\natural$ denoting either the label u or s, by counting reducible instantons whose

$s \to -\infty$ and $s \to \infty$ limits are in $\mathfrak{C}^\sharp$ and $\mathfrak{C}^\sharp$ respectively, with $\bar{\mathrm{gr}}$ differ by -1. If the Spin^c-structure and $[\varpi]$ satisfy monotonicity condition, then the differentials for the complexes, $\hat\partial\colon \hat C \to \hat C$, $\bar\partial\colon \bar C \to \bar C$, $\check\partial\colon \check C \to \check C$ are defined in terms of these homomorphisms via Equation (22.7) and Definition 22.1.3 in [KM]. To give an example, $\hat\partial\colon C^o \oplus C^u \to C^o \oplus C^u$ is written in block form as:

$$\begin{bmatrix} \partial^o_o & -\partial^u_o\bar\partial^s_u \\ \partial^o_s & \bar\partial^s_s - \partial^u_s\bar\partial^s_u \end{bmatrix}. \tag{2.14}$$

The gluing theorems in [KM] show that $\hat\partial^2$, $\bar\partial^2$, $\check\partial^2$ are indeed all 0. When the perturbation is balanced, such as in the statement of Theorem 1.1, the homology of these chain complexes $(\mathring{C}_*, \mathring{\partial}_*)$, namely the corresponding monopole Floer homology, is denoted $\mathring{HM}_*(M, \mathfrak{s}, c_b)$ for $\circ = \wedge, -, \vee$.

The aforementioned homomorphisms $\partial^\sharp_\natural$, $\bar\partial^\sharp_\natural$ are also used to define chain maps (denoted $i\colon \bar C \to \check C$, $j\colon \check C \to \hat C$, $p\colon \hat C \to \bar C$ in [KM]) that do not define a short exact sequence, but their induced maps on homologies do, this being the first of the fundamental exact sequences referred to in Theorem 1.1. See Proposition 22.2.1 in [KM].

In the most general case, one typically needs to work with Floer homology with local coefficients which satisfy certain completeness conditions depending on the choice of $\mathfrak{s}$ and $[\varpi]$. See Definition 30.2.2 in [KM]. We call a local system Γ satisfying this completeness condition $(\mathfrak{s}, [\varpi])$-*complete* (as opposed to "c-complete" in [KM]). There is also a more stringent notion of completeness which depends only on the cohomology class $[d\mathfrak{a}]$ due original to Novikov. This sort of local system is said to be "strongly c-complete" in [KM]; see Definition 30.2.4 therein. We call such Γ *strongly* $(\mathfrak{s}, [\varpi])$-*complete* instead. We shall not encounter such local systems except in Proposition 6.2 (b) below, which is not directly relevant to the proof of Theorem 1.1. The interested reader is therefore referred to [KM] for the definition of Floer homology with local coefficients. A brief summary in alternative language may also be found in the last section of [LT]. The notation

$$\mathring{C}_*(M, \mathfrak{s}, [\varpi]; \Gamma), \quad \circ = \wedge, -, \vee,$$

is used to denote the monopople Floer complex corresponding to (2.9) with an $(\mathfrak{s}, [\varpi])$-complete local coefficients Γ, and $\mathring{HM}_*(M, \mathfrak{s}, [\varpi]; \Gamma)$ for the corresponding monopole Floer homology. In particular, when $[\varpi] = 2\pi c_1(\det \mathbb{S})$, $\mathring{C}_*(M, \mathfrak{s}, [\varpi]; \Gamma)$ and $\mathring{HM}_*(M, \mathfrak{s}, [\varpi]; \Gamma)$ are also respectively denoted by $\mathring{C}_*(M, \mathfrak{s}, c_b; \Gamma)$ and $\mathring{HM}_*(M, \mathfrak{s}, c_b; \Gamma)$.

The following (admittedly sloppy) convention will be adopted for the rest of this article: Since the Floer complexes $(\check C, \check\partial) = (\hat C, \hat\partial) = (C^o, \partial^o_o)$ when the perturbation

is non-balanced, we use CM or (CM, ∂) to denote the one complex. When we wish to emphasize the Spin^c manifold and/or cohomology class of perturbation used to define the monopole Floer complex, these data are added to the above expression in parentheses such as $CM_*(M, \mathfrak{s}, [\varpi])$ or $\big(CM_*(M, \mathfrak{s}), \partial_*(M, \mathfrak{s})\big)$.

2.6 Cobordism-induced chain maps between monopole Floer complexes

Instantons on cobordisms X described in Section 2.2 above are used to define chain maps between the monopole Floer complexes. Details of the construction of these maps are given in Chapter VII of [KM] for cobordisms X between *connected* 3-manifolds Y_- and Y_+, even though properties of moduli spaces of Seiberg-Witten instantons on more general X, where Y_- may be disconnected, are also established therein. In particular, taking X to be a product $\mathbb{R} \times M$, this construction is used to define chain maps from $\mathring{C}$ back to itself, $\circ = \wedge, -, \vee$, that induce the $\mathbf{A}_\dagger$-module structure on the corresponding Floer homology. These chain maps, which generalized the maps U, $\mathfrak{t}_i$ in the non-balanced case, shall be denoted by $\mathring{U}\colon \mathring{C} \circlearrowleft$ and $\mathring{\mathfrak{t}}_i\colon \mathring{C} \circlearrowleft$. As an example of their explicit formulae, $\hat{U}\colon C^o \oplus C^u \to C^o \oplus C^u$ is given in block form as:

$$\begin{bmatrix} U^o_o & U^u_o \\ \bar{U}^s_u \partial^o_s - \bar{\partial}^s_u U^o_s & \bar{U}^u_u + \bar{U}^s_u \partial^u_s - \bar{\partial}^s_u U^u_s \end{bmatrix}.$$

Another application of these cobordism-induced chain maps is to define chain homotopies of $\mathbf{A}_\dagger$-modules between monopole Floer complexes $\mathring{C}$ associated to different metrics and $(\mathfrak{T}, \mathfrak{S})$. See e.g. the proof for Corollary 23.1.6 in [KM] and its variants. According to the conventions set forth in Section 1.3, this justifies our notation for the monopole Floer complex, $\mathring{C}(M, \mathfrak{s}, [\varpi]; \Gamma)$, introduced in the previous subsection. In fact, this type of arguments show that $\mathcal{C}$ with positively proportional "period class" ([KM] p.591) are chain homotopic to each other. (See Theorem 31.4.1 in [KM]). This justifies using the notation $CM(M, \mathfrak{s}, c_-)$ for any negatively monotone perturbation, according to our convention.

The rest of this subsection is divided into two parts. The first part consists of a brief summary of the construction to the afore-mentioned cobordism-induced maps; the second part contains a simple generalization of [KM]'s construction to accomodate our needs in Section 6.

Part 1: Fix a Spin^c structure $\mathfrak{s}_X$ on X which restricts to the $s \leq -2$ and $s \geq 2$ part of X respectively as Spin^c structures $\mathfrak{s}_-$ on Y_- and $\mathfrak{s}_+$ on Y_+. Fix also a self-dual two form ϖ_X on X satisfying (2.10) and a suitable pair $(\mathfrak{T}^+, \mathfrak{S}^+)$. Let $c_{\mathfrak{s}_X}$ denote

the divisibility of $c_1(\mathfrak{s}_X)$. This number divides both $c_{\mathfrak{s}_-}$ and $c_{\mathfrak{s}_+}$. Consider instantons $\mathfrak{d}$ defined from (2.9) with a fixed representative of $\mathfrak{c}_-$ as its $s \to -\infty$ limit and an element in the gauge orbit of $\mathfrak{c}_+$ as its $s \to \infty$ limit. The index of the Fredholm operator that entered the definition of nondegeneracy for instantons is denoted by $\imath_{\mathfrak{d}}$. This generalizes the index in the case of product cobordisms described in the previous subsection; see again Chapter 24 of [KM].

Let $\mathcal{M}_k(X; \mathfrak{c}_-, \mathfrak{c}_+)$ denote the space of such instantons with $\imath_{\mathfrak{d}} = k$. These spaces are to be oriented according to the rules specified in [KM]. All these spaces are in the orbit space of $\mathcal{C}^\sigma(X)$ under the gauge action. Let $\mathcal{U}$ denotes a covering of the latter space satisfying certain transversality condition associated to $\bigcup_{d\leq k} \mathcal{M}_d(X; \mathfrak{c}_-, \mathfrak{c}_+)$ (Cf. Chapter 21 in [KM]); and let $u \in C^k(\mathcal{U}; \mathbb{K})$ be a Čech cochain associated to $\mathcal{U}$. For each fixed Spin^c-structure, introduce homomorphisms $m^\sharp_\natural(u, \mathfrak{s}_X)\colon C^\sharp(Y_-, \mathfrak{s}_-) \to C^\natural(Y_+, \mathfrak{s}_+)$ for $\sharp = o, u$, $\natural = o, s$, by the rule

$$\mathfrak{C}^\sharp \ni \mathfrak{c}_- \mapsto \sum_{\mathfrak{c}_+ \in \mathfrak{C}^\natural} \langle\, u, \mathcal{M}_k(X; \mathfrak{c}_-, \mathfrak{c}_+) \,\rangle\, \mathfrak{c}_+. \tag{2.15}$$

The numbers $\langle\, u, \mathcal{M}_k(X; \mathfrak{c}_-, \mathfrak{c}_+)\,\rangle$ can be defined only if $\mathcal{M}_k(X; \mathfrak{c}_-, \mathfrak{c}_+)$ has certain compactness properties. As with $\partial^\sharp_\natural$, these are present if $\mathfrak{s}_X$ or ϖ_X satisfy certain constraints, or if local coefficients are used. Lemma 25.3.1 in [KM] guarantees that the constraints for the former case are met when

$$\begin{aligned} &\varpi_X = 2\mathrm{r}w_X \text{ for r} \neq 0 \text{ and a } w_X \text{ satisfying (2.11)},\\ &\text{and when } X_{tor} \neq s^{-1}(\mathbb{R} - \{0\}). \end{aligned} \tag{2.16}$$

A companion homomorphism to $m^\sharp_\natural$, denoted $\bar{m}^\sharp_\natural$, is defined using moduli spaces of *reducible* instantons in place of $\mathcal{M}_k(X; \mathfrak{c}_-, \mathfrak{c}_+)$ when the labels $\sharp$, $\natural$ are either u or s. This $\bar{m}^\sharp_\natural$ is the analog of the homomorphism $\bar{\partial}^\sharp_\natural$ from the previous subsection. Once in place, $m^\sharp_\natural$, $\bar{m}^\sharp_\natural$, $\partial^\sharp_\natural(Y_\pm, \mathfrak{s}_\pm)$ can be assembled according to the formulae in (25.5) and Definition 25.3.3 in [KM] into homomorphisms $\mathring{m}[u](X, \mathfrak{s}_X)\colon \mathring{C}_*(Y_-, \mathfrak{s}_-) \to \mathring{C}_*(Y_+, \mathfrak{s}_+)$ for $\circ = \vee, -, \wedge$. For exampe, for $u \in C^k(\mathcal{U})$, $\hat{m}[u]\colon C^o(Y_-) \oplus C^u(Y_-) \to C^o(Y_+) \oplus C^u(Y_+)$ is given in block form as:

$$\begin{bmatrix} m^o_o[u] & m^u_o[u] \\ (-1)^k \bar{m}^s_u[u]\partial^o_s - \bar{\partial}^s_u m^o_s[u] & (-1)^k \bar{m}^u_u[u] + (-1)^k \bar{m}^s_u \partial^u_s - \bar{\partial}^s_u m^u_s[u] \end{bmatrix}.$$

The gluing theorems in Section 24.7 of [KM] show that these are chain maps when both $\mathring{C}_*(Y_-, \mathfrak{s}_-)$ and $\mathring{C}_*(Y_+, \mathfrak{s}_+)$ are regarded as chain complexes with relative $\mathbb{Z}/c_{\mathfrak{s}_X}$-grading.

To have the maps between Floer homologies induced by these chain maps to be behave well when composing cobordisms (exemplified by Proposition 23.2.2 in [KM]), one work with the assembled maps

$$\mathring{m}[u](X) = \sum_{\mathfrak{s}_X} \mathring{m}[u](X, \mathfrak{s}_X) \colon \bigoplus_{\mathfrak{s}_-} \mathring{C}_*(Y_-, \mathfrak{s}_-) \to \bigoplus_{\mathfrak{s}_+} \mathring{C}_*(Y_+, \mathfrak{s}_+), \quad \circ = \vee, -, \wedge,$$

where the direct sum $\bigoplus_{\mathfrak{s}_\pm}$ is over the set of all Spin^c structures on $Y_\pm$, and $\mathfrak{s}_X$ runs through all Spin^c structures on X. As explained in Remark 24.6.6 in [KM], there can be infinitely many $\mathfrak{s}_X$ to sum over for a fixed pair of $\mathfrak{s}_-$, $\mathfrak{s}_+$. This necessitates the replacement of the chain complexes $\mathring{C}_*(Y_-, \mathfrak{s}_-)$, $\mathring{C}_*(Y_+, \mathfrak{s}_+)$ in the preceding expression by their "grading-completed" variants, $\mathring{C}_\bullet(Y_-, \mathfrak{s}_-)$, $\mathring{C}_\bullet(Y_+, \mathfrak{s}_+)$ (Cf. Definition 3.1.3 and paragraphs around (30.1) in [KM]). The cobordisms relevant to our proof of Theorem 1.1 however have $H^2(X, Y_-) = 0$, and this is why we use of the pre-completion Floer complexes $\mathring{C}_*$ as the domain and target of $\mathring{m}[u]$.

As this article involves only minimal reference to Floer homologies with local coefficients, we shall skip details of general cobordism-induced maps between Floer complexes with local coefficients. It suffices to say here that they depend on $(X, \mathfrak{s}_X)$, the 2-form ϖ_X used in the Seiberg-Witten equations (2.9), and a so-called "X-morphism" Γ_X between local systems Γ_- on $\mathcal{B}^\sigma(Y_-)$ and Γ_+ on $\mathcal{B}^\sigma(Y_+)$. ([KM] Definition 23.3.1). In general the local systems must satisfy certain completeness conditions, and Γ_X depends on the relative homotopy class of $\mathfrak{d} \in \mathcal{M}(X, \mathfrak{c}_-, \mathfrak{c}_+) \subset \mathcal{B}^\sigma(X, \mathfrak{c}_-, \mathfrak{c}_+)$ in (2.15). Namely, the connected component that $\mathfrak{d}$ belongs to in the space consisting of $(\mathbb{A}, \Psi)$ on X with $s \to -\infty$ on a representative of $\mathfrak{c}_- \in \mathcal{B}^\sigma(Y_-)$ and $s \to \infty$ limit in the orbit of $\mathfrak{c}_+$. We denote this class by $[\mathfrak{d}]$.

More details will be given on a case-by-case basis as occasions arise. See also Section 25.3 in [KM] contains some discussion on the case with $\varpi_X = 0$.

Part 2: The proof of Theorem 1.1 also requires chain maps associated to more general cobordisms. For this purpose, it suffices to consider the $\hat{m}$ variant of the chain map for cobordisms X satisfying the following constraint:

(2.17) At most one of Y_- or Y_+ is disconnected, in which case it consists of two components. Moreover, at most one end of X is associated with balanced perturbation.

Assume that one of Y_-, Y_+ is of the form $Y_\sqcup = Y_1 \sqcup Y_2$ for connected Y_1 and Y_2, while the other is connected. Take $Y_- = Y_\sqcup$ for example, since the case where $Y_+ = Y_\sqcup$ is entirely parallel. Given the self-dual 2-form ϖ_X described in (2.10), We shall always take Y_2 to be the only end of X possibly associated with a balanced perturbation. Thus, $\mathfrak{C}(Y_\sqcup) = \mathfrak{C}(Y_1) \times \mathfrak{C}(Y_2) = \mathfrak{C}^{oo} \sqcup \mathfrak{C}^{ou} \sqcup \mathfrak{C}^{os}$, with $\mathfrak{C}^{oo}$, $\mathfrak{C}^{ou}$, $\mathfrak{C}^{os}$ denoting $\mathfrak{C}^o(Y_1) \times \mathfrak{C}^o(Y_2)$, $\mathfrak{C}^o(Y_1) \times \mathfrak{C}^u(Y_2)$, $\mathfrak{C}^o(Y_1) \times \mathfrak{C}^s(Y_2)$ respectively. Let $C^{oo}(Y_\sqcup) = \mathbb{K}(\mathfrak{C}^{oo}) = CM(Y_1) \otimes C^o(Y_2)$, $C^{ou}(Y_\sqcup) = \mathbb{K}(\mathfrak{C}^{ou}) = CM(Y_1) \otimes C^u(Y_2)$, $C^{os}(Y_\sqcup) = \mathbb{K}(\mathfrak{C}^{os}) = CM(Y_1) \otimes C^s(Y_2)$.

Take $(\hat{C}(Y_\sqcup), \hat{\partial}_{Y_\sqcup})$ to be the product complex of $(CM(Y_1), \partial(Y_1))$ and $(\hat{C}(Y_2), \hat{\partial}(Y_2))$. There is a degree -2 chain map $\hat{U}_{Y_\sqcup}\colon (\hat{C}(Y_\sqcup), \partial_{Y_\sqcup}) \to (\hat{C}(Y_\sqcup), \partial_{Y_\sqcup})$ given by

$$\hat{U}_{Y_\sqcup} = \hat{U}_{Y_1} \otimes 1 - 1 \otimes \hat{U}_{Y_2}. \tag{2.18}$$

In these cases we have the analogs of $m^\sharp_\natural$ in [KM], this being the homomorphisms $m^{o\sharp}_\natural\colon CM(Y_1) \otimes C^\sharp(Y_2) \to C^\natural(Y_+)$ (or in the case where $Y_\sqcup = Y_+$, $m^\sharp_{o\natural}\colon C^\sharp(Y_-) \to CM(Y_1) \otimes C^\natural(Y_2)$), with $\sharp$ standing for o or u; and with the label $\natural$ standing for o or s. As the condition (2.17) implies that $\hat{C}(Y_\#) = CM(Y_\#)$, $\hat{C}(Y_\sqcup) = C^{oo} \oplus C^{ou}$, the maps $\bar{m} = 0$, $\hat{m}[u]\colon C^{oo} \oplus C^{ou} \to CM(Y_\#)$ and $\hat{m}[u]\colon CM(Y_\#) \to C^{oo} \oplus C^{ou}$ respectively take the following simple form:

$$\begin{bmatrix} m^{oo}_o & m^{ou}_o \end{bmatrix}, \quad \begin{bmatrix} m^o_{oo} \\ -(1 \otimes \bar{\partial}^s_u(Y_2)) \circ m^o_{os} \end{bmatrix}. \tag{2.19}$$

Lastly, parallel to the final remarks in Section 2.3, we note that a cobordism X determines a relation $\sim_X$ between the grading sets $\mathbb{J}(Y_-)$ and $\mathbb{J}(Y_+)$ mentioned in 2.3. The algebra $\mathbf{A}_\dagger(X) := \bigwedge^*(H_1(X)/\operatorname{Tors}) \otimes \mathbb{K}[U]$ acts on $\operatorname{Hom}(\mathring{C}(Y_-), \mathring{C}(Y_+))$ and interwines with the $\mathbf{A}_\dagger(Y_-)$-action on $\mathring{C}(Y_-)$ and $\mathbf{A}_\dagger(Y_+)$-action on $\mathring{C}(Y_+)$. (Cf. p.76 of [KM]). There is also a notion of *homology orientation* for cobordisms which determines the orientations of the moduli spaces in (2.15). See Definition 3.4.1 in [KM]. Generalization of these notions in the situation described in Part 2 above will be addressed minimally in Section 6.1.

3 Filtered monopole Floer homologies

The algebraic recipe for Ozsvath-Szabo's definition of the four flavors of Heegaard Floer homologies, labeled by the superscripts $-, \infty, +, \wedge$, was summarized abstractly in section 4 of [L]. In this section, we explain how the same recipe may be applied in the Seiberg-Witten context to define analogs of Ozsvath-Szabo's Floer homologies. These intermediate Floer homologies play a pivotal role in the proof of Theorem 1.1.

3.1 Motivation and sketches of construction

The afore-mentioned recipe hinges on the existence of certain filtration on a Floer chain complex with local coefficients in the group ring $\mathbb{K}[\mathbb{Z}] = \mathbb{K}[U, U^{-1}]$, with U corresponding to the generator $1 \in \mathbb{Z}$. This Floer complex with local cofficients constitutes the ∞-flavor of the Ozsvath-Szabo construction, while the "filtration" refers to the filtration of the coefficient ring $\mathbb{K}[U, U^{-1}]$ by submodules

$$\cdots U\,\mathbb{K}[U] \subset \mathbb{K}[U] \subset U^{-1}\mathbb{K}[U] \subset \cdots \subset \mathbb{K}[U, U^{-1}].$$

If the differential of the ∞-flavor of the Floer complex preserves this filtration, then it induces a filtration on the ∞-flavor Floer complex by $\mathbb{K}[U]$-subcomplexes, which are all isomorphic via multiplication by powers of U. This defines the $-$-flavor Floer complex. With these two basic flavors in place, the $+$- and the $\wedge$-flavors are defined so that they fit into short exact sequences (see (3.16) below) inducing what are called the *fundamental exact sequences* of corresponding Floer homologies. In [L], the existence of such filtration is attributed to the existence of what was termed a "semi-positive 1-cocycle". The 1-cocycle used here refers to the cocycle that defines the local system on the ∞-flavor of Floer complex. The "semi-positivity" condition serves to guarantee that the differential is filtration-preserving. Note that the ∞-flavor of Floer homology depends only on the cohomology class of this cocycle. The other three flavors of Ozsvath-Szabo's construction depend on the choice the cocycle that defines the semi-positivity condition.

Section 4.2 of [L] provides some examples where this recipe may be applied. Section 6 of the same article sketched how such semi-positive 1-cocycles might arise in certain versions of Seiberg-Witten Floer theory associated to equations of the form of (2.4). In particular, choosing the metric and 2-form w in (2.4) to reflect the data that go into the definition of Heegaard Floer homology provides a bridge to relate the Heegaard and Seiberg-Witten Floer homologies.

To elaborate, the local system underlying the Seiberg-Witten analog of Ozsvath-Szabo construction is closely related to what was denoted Γ_η in [KM] (Cf. Example in the end of their Section 22.6), where η is a singular 1-cycle in a certain 3-manifold $\underline{M}$. Use $[(\mathbb{A}, \Psi)] \in \mathcal{B}^\sigma$ to denote the gauge equivalence of $(\mathbb{A}, \Psi)$. In [KM], this local system associates to each point on $\mathcal{B}^\sigma$ the "fiber" $\mathbb{R}$, and to each path $\{[(\mathbb{A}(\tau), \Psi(\tau))]\}_\tau$ from $[(\mathbb{A}_-, \Psi_-)]$ to $[(\mathbb{A}_+, \Psi_+)]$, an isomorphism $\mathbb{R}^\times \subset \mathrm{End}(\mathbb{R})$ between the fibers over the end points. The latter isomorphism is given by multiplication by the real number

$$e^{\frac{i}{2\pi}\int_\tau \int_\eta \frac{d}{d\tau}\mathbb{A}(\tau)}. \tag{3.1}$$

Note that the exponent is the difference of the holonomy of $\mathbb{A}_-$ along the cycle $\eta \subset M$ from that of $\mathbb{A}_+$, and it defines a real 1-cocycle in $\mathcal{B}^\sigma$. Meanwhile, as only points in $\mathfrak{C} \subset \mathcal{B}^\sigma$ and paths constituting the sets $\mathcal{M}_1(\mathfrak{c}_-, \mathfrak{c}_+)$, $\mathfrak{c}_-$, $\mathfrak{c}_+ \in \mathfrak{C}$ enter the definition of a monople Floer complex, it suffices to consider the holonomy difference of paths corresponding to elements in $\mathcal{M}_1(\mathfrak{c}_-, \mathfrak{c}_+)$. The observation leading to [L]'s construction of filtered monopole Floer homologies (in the sense of Ozsvath-Szabo) is the following:

(3.2) for monopole Floer complexes associated to certain ϖ in the form of (2.3) with large r and certain choice of η, the value of the afore-mentioned holonomy difference is very close to a non-negative integer.

Associating to each element in $\mathcal{M}_1(\mathfrak{c}_-, \mathfrak{c}_+)$ its corresponding integer, one has a (partially defined) integer 1-cocycle on $\mathcal{B}^\sigma$ with which one may define a Floer complex with more refined local coefficients than Γ_η. We denote the latter local system by Λ_η. It replaces the fibers $\mathbb{R}$ over $\mathfrak{C}$ of Γ_η by the group ring $\mathbb{K}[\mathbb{Z}] = \mathbb{K}[U, U^{-1}]$; and it replaces the isomorphism induced by an element $\mathfrak{d}$ in $\mathcal{M}_1(\mathfrak{c}_-, \mathfrak{c}_+)$ between these fibers, namely (3.1), by U^n where n denotes the afore-mentioned non-negative integer associated to $\mathfrak{d}$. The fact that $n \geq 0$ in all cases has the following consequence: Use the corresponding monopole Floer complex with local coefficients, Λ_η, as the ∞*-flavor Floer complex*. There is filtration on this chain complex, $CM_*(\underline{M}; \Lambda_\eta)$, by subcomplexes of $\mathbb{K}[U]$-modules. This can be used to define the other three flavors of Floer complexes.

The program described in [L] assumes various plausible conjectures and assertions that come from an extension of the geometric picture in the last author's work relating the Seiberg-Witten and Gromov invariant for closed 4-manifolds (Cf. [T], [Ta1]). A proof of these conjectures constitute a major part of the technical hurdle for implementing the program in [L]. The difficulties arise because the 2-form ϖ in [L] must have zeros.

In this series of articles [KLT1]-[KLT4], the road block to the approach in [L] is circumvented by a modification of [L]'s outline. Very roughly, the manifold $\underline{M}$ in [L] is replaced by the manifold denoted by Y in [KLT2]. This is obtained from $\underline{M}$ by adding further 1-handles along the zeros of w on $\underline{M}$. The 2-form w extends into Y as a no-where vanishing closed 2-form, which we also denote by w. Over the middle of the added 1-handle, this w approximates da for a certain contact form a, and as the special 1-cycle η (denoted $\underline{\gamma}$ therein) lies away from the zeros of w on $\underline{M}$, this 1-cycle also embeds in Y. This was denoted by $\gamma^{(z_0)}$ in [KLT1]-[KLT4]. The technical challenge in this new approach involves, among other things, the analog of (3.2) for the monopole Floer complex associated to Y, w, and $\eta = \gamma^{(z_0)}$. Some of these technical

issues are dealt with in [KLT4]. Those that remain are dealt with in Section 7-9 of this article.

In Section 3.2 below, we specify the class of 3-manifolds, denoted Y_Z therein, together with the 2-form w on it and the 1-cycle η for which statements of the kind (3.2) hold. Section 3.3 describes the sort of cobordisms X for which the companion statements hold. The remaining subsections give precise statements of the desired positivity results. The conditions on Y_Z and X are introduced more for technical convenience rather than essential reasons, and the statements in Sections 3.4-3.7 may conceivably hold for more general 3-manifolds and 4-dimensional cobordisms.

3.2 The 3-manifold Y_Z

Let Z denote a given connected, oriented closed 3-manifold; and let Y_Z denote the manifold that is obtained from Z by attaching a 1-handle at a chosen pair of points, denoted (p_0, p_3) below. In the proof of the main theorem 1.1, Z is taken to be either S^3, the manifold M in the statement of Theorem 1.1, or a manifold that is obtained from M by attaching some number of 1-handles. Although Y_Z is diffeomorphic to the connected sum of Z and $S^1 \times S^2$, it is viewed for the most part as $Z_\delta \cup \mathcal{H}_0$ with $\mathcal{H}_0$ the attached 1-handle and with Z_δ being the complement of a pair of coordinate balls about the chosen points p_0 and p_3 in Z. The manifold Y_Z has a distinguished embedded loop that crosses the handle $\mathcal{H}_0$ once. This loop is denoted by γ. The three parts of this subsection say more about the geometry of Y_Z near $\mathcal{H}_0$, near γ, and in general.

Part 1: The geometry of Y_Z near $\mathcal{H}_0$ is just like that given in Section II.1a. By way of a reminder, the description of the geometry requires the a priori specification of constants $\delta_* \in (0, 1)$ and $R > -100 \ln \delta_*$. Also needed are coordinate charts centered on p_0 and p_3. The latter are used to identify respective neighborhoods of these points with balls of radius $10\delta_*$ in $\mathbb{R}^3$. The pull-back of the standard spherical coordinates on $\mathbb{R}^3$ gives spherical coordinate functions on the neighborhood of p_0, these denoted by $(\mathrm{r}_+, (\theta_+, \phi_+))$. There are corresponding coordinate functions for the neighborhood of p_3; these are denoted in what follows by $(\mathrm{r}_-, (\theta_-, \phi_-))$.

The handle $\mathcal{H}_0$ is diffeomorphic to the product of an interval with S^2. The interval factor is written as $[-R - 7 \ln \delta_*, R + 7 \ln \delta_*]$ and u is used to denote the Euclidean coordinate for this interval. The spherical coordinates for the S^2 factor are written as (θ, ϕ). The handle $\mathcal{H}_0$ is attached to the coordinate balls centered on p_0 and p_3 as

follows: Delete the $\mathrm{r}_+ < e^{-2R}(7\delta_*)^{-1}$ part the coordinate ball centered on p_0 and the corresponding part of the coordinate ball centered on p_3. Having done so, identify $\mathcal{H}_0$ with the respective $\mathrm{r}_+ \in [e^{-2R}(7\delta_*)^{-1}, 7\delta_*]$ and $\mathrm{r}_- \in [e^{-2R}(7\delta_*)^{-1}, 7\delta_*]$ parts of these coordinate balls with $\mathcal{H}_0$ by writing

$$(3.3)\qquad \begin{aligned} &(\mathrm{r}_+ = e^{-R+u}, (\theta_+ = \theta, \phi_+ = \phi)) \quad \textit{and} \\ &(\mathrm{r}_- = e^{-R-u}, (\theta_- = \pi - \theta, \phi_- = \phi)). \end{aligned}$$

The handle $\mathcal{H}_0$ has a distinguished closed 2-form, this being $\frac{1}{2}\sin\theta d\theta d\phi$. This 2-form is nowhere zero on the constant u cross-sectional spheres and thus orients these spheres. Granted this orientation, then $\frac{1}{2}\sin\theta d\theta d\phi$ has integral 2 over constant u sphere.

Part 2: The loop γ intersects $\mathcal{H}_0$ as the $\theta = 0$ arc. Thus it has geometric intersection number 1 with each $u =$ constant sphere. This loop is oriented so that the corresponding algebraic intersection number is $+1$. A tubular neighborhood of γ is specified with a diffeomorphism to the product of S^1 and a disk about the origin in $\mathbb{C}$. The latter is denoted by D_γ and its complex coordinate is denoted by z. The diffeomorphism identifies the $z = 0$ circle in $S^1 \times D$ with γ. The circle S^1 is written in what follows as $\mathbb{R}/(\ell_\gamma\mathbb{Z})$ with $\ell_\gamma > 0$ being a chosen constant. The affine coordinate for $\mathbb{R}/(\ell_\gamma\mathbb{Z})$ is denoted by t. The product structure on such a neighborhood is constrained where it intersects $\mathcal{H}_0$ by the requirement that the $\mathcal{H}_0$ coordinate u on the intersection depend only on t. A neighborhood with these coordinates is fixed once and for all; it is denoted by U_γ.

Part 3: Use the Mayer-Vietoris principle to write the second homology of Y_Z as

$$(3.4)\qquad H_2(Y_Z;\mathbb{Z}) = H_2(Z;\mathbb{Z}) \oplus H_2(\mathcal{H}_0;\mathbb{Z}).$$

The convention in what follows is to take the generator of $H_2(\mathcal{H}_0;\mathbb{Z})$ to be the class of any cross-sectional sphere with the orientation given by the 2-form $\sin\theta d\theta d\phi$. Fix a class in $H^2(Y_Z;\mathbb{Z})$ which has even pairing with the classes in $H_2(Y_Z;\mathbb{Z})$ and pairing 2 with the generator of the $H_2(\mathcal{H}_0;\mathbb{Z})$ summand in (3.4). This class is denoted in what follows by $c_1(\det(\mathbb{S}))$, and it is necessarily non-torsion by the above assumption.

There is a corresponding, closed 2-form on Y_Z whose de Rham cohomology class is that of $c_1(\det(\mathbb{S}))$. In particular, there are forms w of this sort satisfying the following

additional constraints:

(3.5)
- The form restricts to $\mathcal{H}_0$ as $\frac{1}{2\pi}\sin\theta\, d\theta\, d\phi$;
- The form restricts to U_γ as $\frac{i}{2\pi}g(|z|)\, dz \wedge d\overline{z}$ with g denoting a strictly positive function.
- There is a closed 1-form on Y_Z, typically denoted by υ below, with the following properties:
  a) It has non-negative wedge product with w.
  b) It restricts to U_γ as dt, and restricts to $\mathcal{H}_0$ as $\mathrm{H}(u)\, du$ with $\mathrm{H}(u) > 0$ for all u.

Fix such a 2-form as the perturbation form w in (2.4).

The metric on Y_Z is chosen to satisfy the following constraints:

(3.6)
- The metric appears on $\mathcal{H}_0$ as the product metric of an S^2-independent metric on the interval $[-R - \ln(7\delta_*), R + \ln(7\delta_*)]$ and the round metric $d\theta^2 + \sin^2\theta d\phi^2$ on the S^2 factor. Meanwhile, the curvature 2-form of A_K on $\mathcal{H}_0$ is $\frac{i}{2\pi}\sin\theta\, d\theta\, d\phi$.
- The metric appears on U_γ as $dt^2 + g(|z|)\, dz \otimes d\overline{z}$ with g being the function in the second bullet of (3.5). Meanwhile, A_K has holonomy 1 on γ and its curvature 2-form on U_γ is iw.

Many of the lemmas and propositions in the rest of this section depend implicitly on the radius of D_{r} and on the injectivity radius of the Riemannian metric. They also depend implicitly on the norms of w, the curvature of A_K, the Riemannian curvature, and the norms of their derivatives up to some order less than 10.

There are suitable choices for μ with positive but small as desired $\mathcal{P}$-norm that vanish on $\mathcal{H}_0 \cup U_\gamma$. This last property is not a direct consequence of an explicit assertion in [KM] but it follows nonetheless from their constructions.

The connection A_E is chosen constrained only to the extent that it is flat on $\mathcal{H}_0$ and is flat with holonomy 1 on U_γ.

The function on $\mathrm{Conn}\,(E) \times C^\infty(Y_Z; \mathbb{S})$ of central concern in what follows is the analog here of the function that is defined in (IV.1.16). This function is denoted by X. The definition requires the a priori choice of a smooth function $\wp : [0,\infty) \to [0,\infty)$ which is non-decreasing, obeys $\wp(x) = 0$ for $x < \frac{7}{16}$ and $\wp(x) = 1$ for $x \geq \frac{9}{16}$. As in [KLT4], it proves convenient to choose $\wp$ so that its derivative, $\wp'$, is bounded by $2^{10}(1-\wp)^{3/4}$. The definition of X uses the fact that w is nowhere zero on U_γ. In particular, Clifford

multiplication by $*w$ on U_γ splits $\mathbb{S}$ over U_γ as the direct sum of eigenbundles. This splitting is $\mathbb{S} = E \oplus E \otimes K^{-1}$ with the convention being that $*w$ acts as $+i|w|$ on E. A given section ψ of $\mathbb{S}$ is written with respect to this splitting over U_γ as a pair denoted by $|w|^{1/2}(\alpha, \beta)$.

Granted this notation, use $\wp$ with a given pair $\mathfrak{c} = (A, \psi) \in \mathrm{Conn}\,(E) \times C^\infty(Y_Z; \mathbb{S})$ to define the connection

$$\hat{A} = A - \wp(|\alpha|^2)|\alpha|^{-2}(\bar{\alpha}\nabla_A\alpha - \alpha\nabla_A\bar{\alpha}), \tag{3.7}$$

on $E|_{U_\gamma}$. The value of $\mathrm{x} = \mathrm{x}_\gamma$ on $\mathfrak{c}$ is

$$\mathrm{x}(\mathfrak{c}) = \frac{i}{2\pi}\int_\gamma(\hat{A} - A_E). \tag{3.8}$$

The following lemma supplies a fundamental observation about x.

Lemma 3.1 *If the conditions in (3.5), (3.6) hold, then there exists $\kappa > \pi$ with the following significance: Fix $\mathrm{r} > \kappa$ and a 1-form $\mu \in \Omega$ with $\mathcal{P}$-norm less than 1. The function x has only integer values on the solutions to the corresponding (r, μ)-version of (2.4).*

This lemma is proved in § 7.3.

3.3 4-dimensional cobordisms

This subsection describes in general terms the sorts of cobordisms that are considered.

To start, let Z_- and Z_+ denote two versions of the manifold Z and let Y_- and Y_+ denote the respective $Z = Z_-$ and $Z = Z_+$ versions of Y_Z. There is no need to assume that either Y_- or Y_+ is connected, but if not, then the handle $\mathcal{H}_0$ is attached to the same connected component. Use γ_- to denote the Y_- version of the curve γ and use γ_+ to denote the Y_+ version. The corresponding versions of U_γ are denoted in what follows by U_- and U_+.

Of interest here is a smooth, oriented, 4-dimensional manifold X with the properties

listed below, in addition to those in (2.7):

(3.9)
- There exists an embedding of $\mathbb{R} \times [-R - \ln(7\delta_*), R + \ln(7\delta_*)] \times S^2$ into X that pulls back s as the Euclidean coordinate on the $\mathbb{R}$-factor. Moreover, the composition of this embedding with the diffeomorphism in the second bullet identifies the $s < 0$ part with $(-\infty, 0) \times \mathcal{H}_0$ in $(-\infty, 0) \times Y_-$; and the composition with the diffeomorphism from the third bullet identifies the $s > 0$ part with $(0, \infty) \times \mathcal{H}_0$ in $(0, \infty) \times Y_+$.
- There exists an embedding of $\mathbb{R} \times S^1$ into X that pulls back s as the Euclidean coordinate on the $\mathbb{R}$-factor. Moreover, the composition of this embedding with the diffeomorphism in the second bullet identifies the $s < 0$ part of $\mathbb{R} \times S^1$ with $(-\infty, 0) \times \gamma_-$; and the composition with the diffeomorphism from the third bullet identifies the $s > 0$ part of $\mathbb{R} \times S^1$ with $(0, \infty) \times \gamma_+$.

The image in X of the embedding of $\mathbb{R} \times [-R - \ln(7\delta_*), R + \ln(7\delta_*)] \times S^2$ from the first bullet above is denoted by U_0.

The notation used in the next constraint has C denoting the image in X of $\mathbb{R} \times S^1$ as described by the second bullet of (3.9). This constraint requires that the γ_- and γ_+ versions of ℓ_γ are equal.

(3.10)
- There exists $\ell_\gamma > 0$ and a diffeomorphism of a neighborhood of C to the product of $\mathbb{R} \times \mathbb{R}/(\ell_\gamma\mathbb{Z})$ with a disk about the origin in $\mathbb{C}$. This disk is denoted by D.
- The diffeomorphism identifies the Euclidean coordinate on $\mathbb{R} \times \mathbb{R}/(\ell_\gamma\mathbb{Z}) \times D$ with s.
- The respective $s < 0$ and $s > 0$ parts of the neighborhood are in $(-\infty, 0) \times U_-$ and in $(0, \infty) \times U_+$. Moreover, the diffeomorphism on these parts of the neighborhood respects the respective splittings of U_- and U_+ as $(-\infty, 0) \times \mathbb{R}/(\ell_\gamma\mathbb{Z}) \times D$ and $(0, \infty) \times \mathbb{R}/(\ell_\gamma\mathbb{Z}) \times D$.

By way of an explanation, a diffeomorphism of this sort exists if the co-normal bundle to C in X has a nowhere zero section that restricts to the $s < 0$ part of X as the real part of the $\mathbb{C}$-valued 1-form dz along γ_- and restricts to the $s > 0$ part of X as the real part of the $\mathbb{C}$-valued 1-form dz along γ_+. The tubular neighborhood in (3.10) is denoted in what follows as U_C. The diffeomorphism in (3.10) is used, often implicitly, to identify U_C with $\mathbb{R} \times \mathbb{R}/(\ell_\gamma\mathbb{Z}) \times D$.

In addition to those listed in (2.11), the 2-form w_X to use in the Seiberg-Witten equations is required to satisfy the following additional constraint:

(3.11)
The pull-back of w_X to U_0 via the embedding from the fourth bullet of (3.9) is twice the self dual part of $\frac{1}{2}\sin\theta d\theta d\phi$ and its pull-back to U_C via the embedding in (3.10) is twice the self dual part of $\frac{i}{2\pi}g(|z|)\,dz\wedge d\bar{z}$.

Meanwhile, the metric on X is required to satisfy the following constraints in addition to those in (2.8):

(3.12)
- The metric pulls back from U_0 via the embedding of the first bullet of (3.9) as the product metric defined by the Euclidean metric on the $\mathbb{R}$-factor and an $\mathbb{R}$-independent product metric on the $[-R-\ln(7\delta_*), R+\ln(7\delta_*)]\times S^2$ factor.
- The metric pulls back from U_C via the embedding in (3.10) as the product metric given by the quadratic form $ds^2+dt^2+g(|z|)\,dz\otimes d\bar{z}$.

Extensions to U_C and U_0 of the Y_- and Y_+ versions of the line bundles K and E and their connections A_K and A_E are needed for what follows. There is no obstruction to making these extensions. Even so, it is necessary to constrain A_K and A_E on Y_- and Y_+ so that extended versions of A_K and A_E on $U_C\cup U_0$ exist with the curvature of the extended version of A_K pulling back via the embeddings from the first bullet of (3.9) and (3.10) as $\sin\theta\,d\theta\,d\phi$ and $g(|z|)\,dz\wedge d\bar{z}$. Meanwhile, the pull-backs of the curvature of A_E via these embeddings is zero. Extensions with this property are assumed implicitly.

The definitions in [KM] are sufficiently flexible so as to allow for the following: For any given $\mathrm{r}>\pi$, there are suitable perturbation terms for (2.9) with positive but as-small-as desired $\mathcal{P}$-norm that vanish on U_C and on the image of $\mathbb{R}\times\mathcal{H}_0$ via the embedding map from the first bullet of (3.9).

With regards to notation and conventions, the propositions and lemmas that follow refer only to (2.9). Even so, all assertions still hold for the versions with an extra perturbation term if the perturbation term has $\mathcal{P}$-norm bounded by $e^{-\mathrm{r}^2}$ or has small, r-independent $\mathcal{P}$-norm and vanishes on U_C and on the image of $\mathbb{R}\times\mathcal{H}_0$ via the embedding from the first bullet of (3.9). Proofs of the propositions and lemmas will likewise refer only to

(2.9). The modifications that are needed to deal with the extra perturbation terms are straightforward and so left to the reader.

The second set of constraints require the choice of constants $c \geq 1$ and $r \geq 1$. By way of notation, one of the upcoming constraints uses the embeddings from the second and third bullets of (2.7) to write w_X on the $|s| \in [L-4, L]$ part of X as $w_X = ds \wedge *w_* + w_*$ with w_* denoting a closed, s-dependent 2-form on Y_- or Y_+, and with $*$ here denoting the Hodge star for the metric $\mathfrak{g}$ in the second bullet of (2.8).

(3.13)

1) The constant L in (2.8) is less than c. The constant L_{tor} in (2.11) is equal to $c \ln r$.
2) The norm of the Riemannian curvature tensor and those of its covariant derivatives up to order 10 are less than $r^{1/c}$ on the $s \in [-L, L]$ part of X.
   - a) The injectivity radius is larger than $r^{-1/c}$ on the $s \in [-L, L]$ part of X.
   - b) The metric volume of the s-inverse image in X of any unit interval is bounded by c.
3) The metric $\mathfrak{g}$ from (2.8)'s second bullet obeys $|\frac{\partial}{\partial s}\mathfrak{g}| \leq r^{1/c}$.
4) The norm of w_X is bounded by c. The norms of its covariant derivatives to order 10 are bounded by $r^{1/c}$ on the $s \in [-L, L]$ part of X.
   - a) The 2-form w_X is closed on the $|s| \leq L - 4$ part of X.
   - b) Use the embeddings from the second and third bullets of (2.7) to write w_X on the $|s| \in [L-4, L]$ parts of $X - X_{tor}$ as $w_X = ds \wedge *w_* + w_*$. Then $\frac{\partial}{\partial s} w_* = dб$ where $б$ is a smooth, s-dependent 1-form on the relevant components of Y_- or Y_+ with $\int_{(X - X_{tor}) \cap |s|^{-1}([L-4, L])} |б|^2 < r^{-1/c}$.
   - c) The 2-form w_X is closed on the components of the $L - 4 \leq |s| \leq L_{tor} - 4$ part of X_{tor}.
   - d) Use the embeddings from the second and third bullets of (2.7) to write w_X on the $|s| \in [L_{tor} - 4, L_{tor}]$ parts of X_{tor} as $w_X = ds \wedge *w_* + w_*$. Then $\frac{\partial}{\partial s} w_* = dб$ where $б$ is a smooth, s-dependent 1-form on the relevant components of Y_- or Y_+ with $\int_{X_{tor} \cap |s|^{-1}([L_{tor}-4, L_{tor}])} |б|^2 < r^{-1/c}$.

(3.13)
$$\begin{cases}
\text{5) There is a smooth, closed 1-form on } X\text{, denoted by } \upsilon_X \text{ below, with norm bounded by } c \text{ and such that:}\\
\quad \text{a) The pull-back of } \upsilon_X \text{ to } (-\infty,-L]\times Y_- \text{ and to } [L,\infty)\times Y_+ \text{ via the embeddings from the second and third bullets of (2.7) is an } s\text{-independent 1-form on } Y_- \text{ and } Y_+.\\
\quad \text{b) The pull back of } \upsilon_X \text{ to } U_C \text{ via the embedding from (3.10) is } dt \text{ and its pull-back to } U_0 \text{ via the embedding from the first bullet of (3.9) is } \mathrm{H}(u)du \text{ with } \mathrm{H}(\cdot)\geq c^{-1}.\\
\quad \text{c) } *(ds\wedge\upsilon_X\wedge w_X)\geq -r^{-1/c} \text{ on the } |s|\in[L-4,\infty) \text{ part of } X.
\end{cases}$$

Definition 3.2 The metric and w_X on X are said to be (c,r)*-compatible* when one of the following conditions are met:

(3.14)
$$\begin{cases}
\bullet\ \text{The space } X=\mathbb{R}\times Y_Z\text{; the metric has the form } ds^2+\mathfrak{g} \text{ with } \mathfrak{g} \text{ being an } s\text{-independent metric on } Y_Z\text{; and the 2-form } w_X \text{ is the } s\text{-independent form } ds\wedge *w+w. \text{ Moreover, there exists a closed 1-form on } Y_Z\text{, denoted by } \upsilon \text{ below, that restricts to } U_\gamma \text{ as } dt\text{, and restricts to } \mathcal{H}_0 \text{ as } \mathrm{H}(u)\,du \text{ with } \mathrm{H}(\cdot)>c^{-1}\text{, and is such that } \upsilon\wedge w\geq -r^{-1/c}.\\
\bullet\ \text{The metric and } w_X \text{ obey the constraints in (2.8), (2.11), (3.11), (3.12) and (3.13).}
\end{cases}$$

By way of a look ahead, the notion of (c,r)-compatibility is invoked below with r given by the constant r in (2.9).

3.4 Positivity on cobordisms

An analog of the connection that is defined in (3.7) plays a role in what follows. This connection is denoted in what follows by $\hat{A}$. To define it, keep in mind that $w_X\neq 0$ on U_C and so Clifford multiplication by w_X^+ on $\mathbb{S}^+$ over U_C or $(-\infty,-2]\times\mathcal{H}_0$ or $[2,\infty)\times\mathcal{H}_0$ splits $\mathbb{S}^+$ as a direct sum of eigenbundles, this written as $\mathbb{S}^+=E\oplus(E\otimes K^{-1})$ with it understood that w_X acts as multiplication by $i|w_X|$ on the left most summand. A section, ψ, of $\mathbb{S}$ is written with respect to this splitting over U_C as $\psi=|w_X|^{1/2}(\alpha,\beta)$. Meanwhile, $\mathbb{A}$ is written as A_K+2A with A being a connection on E. Granted this notation, write $\hat{A}$ using the formula in (3.7) with it understood that the covariant derivatives of α that appear have non-zero pairing with the vector field $\frac{\partial}{\partial s}$. The curvature 2-form of $\hat{A}$ is related to the curvature of A by the rule

$$F_{\hat{A}}=(1-\wp)F_A+\wp'\nabla_A\alpha\wedge\nabla_A\bar{\alpha}. \tag{3.15}$$

With a look ahead at the upcoming propositions, note that the integral of $iF_{\hat{A}}$ over C is well defined when (A,ψ) is an instanton solution to (2.9). This is proved using integration by parts to express the integral of $iF_{\hat{A}}$ as the difference between integrals of the $i\mathbb{R}$-valued 1-form $\hat{A} - A_E$ over respective $s \gg 1$ and $s \ll -1$ slices of C.

The first proposition below concerns the integral of $iF_{\hat{A}}$ on C when X, its metric, and the 2-forms w_X and $\mathfrak{w}_\mu$ define the product cobordism.

Proposition 3.3 *Assume that X, the metric, and w_X can be used to define a product cobordism once μ is chosen. Assume in addition that Y_Z has a closed 1-form, $\upsilon_\diamond$, such that $\upsilon_\diamond \wedge w \geq 0$, whose restriction to U_γ is dt and whose restriction to $\mathcal{H}_0$ is $\mathrm{H}\,du$ with H being a strictly positive function of u. Given $c \geq 1$, there exists $\kappa > \pi$ with the following significance: Fix $\mathrm{r} \geq \kappa$ and $\mu \in \Omega$ with either $\mathcal{P}$-norm bounded by $e^{-\mathrm{r}^2}$ or with $\mathcal{P}$-norm bounded by 1 but vanishing on $\mathbb{R} \times (\mathcal{H}_0 \cup U_\gamma)$. Let $\mathfrak{c}_-$ and $\mathfrak{c}_+$ denote solutions to the (r,μ)-version of (2.4) on Y_Z with $\mathfrak{a}(\mathfrak{c}_-) - \mathfrak{a}(\mathfrak{c}_+) \leq \mathrm{r}^{2-1/c}$. Suppose that $\mathfrak{d} = (\mathbb{A},\psi)$ is an instanton solution to the corresponding version of (2.9) on X with $s \to -\infty$ limit $\mathfrak{c}_-$ and $s \to \infty$ limit $\mathfrak{c}_+$. Then $i\int_C F_{\hat{A}} \geq 0$.*

Proposition 3.3 is a special case of the next proposition which concerns the integral of $iF_{\hat{A}}$ on C when the relevant data does not necessarily define the product cobordism.

Proposition 3.4 *Assume that X and w_X obey the conditions in Sections 3.3, and that the metric on X obeys (2.8) and (3.12). Then there exists $\kappa > \pi$ such that given any $c \geq \kappa$, there exists κ_c with the following property: Fix $\mathrm{r} \geq \kappa_c$ and assume that the metric and w_X are $(c, r = \mathrm{r})$-compatible data. Fix μ_- and μ_+ from the respective Y_-- and Y_+-versions of Ω with either $\mathcal{P}$-norm less than $e^{-\mathrm{r}^2}$ or with $\mathcal{P}$-norm less than 1 but vanishing on the respective Y_-- and Y_+-versions of $\mathcal{H}_0 \cup U_\gamma$. Let $\mathfrak{c}_-$ and $\mathfrak{c}_+$ denote solutions to the (r,μ_-)-version of (2.4) on Y_- and (r,μ_+) -version of (2.4) on Y_+ with $\mathfrak{a}(\mathfrak{c}_-) - \mathfrak{a}(\mathfrak{c}_+) \leq \mathrm{r}^{2-1/c}$. If $\mathfrak{d} = (\mathbb{A},\psi)$ is an instanton solution to (2.9) with $s \to -\infty$ limit $\mathfrak{c}_-$ and $s \to \infty$ limit $\mathfrak{c}_+$, then $i\int_C F_{\hat{A}} \geq 0$.*

Proposition 3.4 is proved in Section 8.2.

3.5 The bound for $\mathfrak{a}(\mathfrak{c}_-) - \mathfrak{a}(\mathfrak{c}_+)$ in Proposition 3.4

Proposition 3.4 concerns only those instanton solutions to (2.9) that obey the added constraint $\mathfrak{a}(\mathfrak{c}_-) - \mathfrak{a}(\mathfrak{c}_+) \leq \mathrm{r}^{2-1/c}$. The two propositions that are stated momentarily are used to guarantee that this constraint is met in the cases that are relevant to the body of this paper. What follows sets the stage for the first proposition.

Definition 3.5 Fix $c > 1$. The metric on Y_Z and the 2-form w are said to define c*-tight* data when there exists a positive, c-dependent constant with the following significance: Use the metric, the 2-form w, a choice of r greater than this constant and a chosen 1-form from Ω with $\mathcal{P}$-norm less than 1 to define (2.4). If $\mathfrak{c}$ is a solution, then $|\mathfrak{a}^{\mathfrak{f}}(\mathfrak{c})| < \mathrm{r}^{2-1/c}$.

Proposition 3.6 *Let Y_Z denote a compact, oriented Riemannian 3-manifold with a chosen Riemannian metric and a* Spin^c*-structure with non-torsion first Chern class. Let w denote a harmonic 2-form on Y_Z whose de Rham class is this first Chern class. Assume that w has non-degenerate zeros on any component of Y_Z where it is not identically zero. Then the metric and w define a c-tight data set if c is sufficiently large.*

This proposition is proved in Section 7.8.

This notion of being c-tight is used in the second of the promised propositions. To set the stage for this one, suppose that X is a cobordism of the sort that is described in Section 3.3. Fix a metric on X and the auxiliary data as described in (2.8), (2.11), (3.11), and let $\mathfrak{d} = (A, \psi)$ denote an instanton solution to a given $\mathrm{r} > \pi$ version of (2.9). Use $\mathfrak{c}_-$ and $\mathfrak{c}_+$ to denote the respective $s \to -\infty$ and $s \to \infty$ limits of $\mathfrak{d}$. Associated to $\mathfrak{d}$ is a certain first order, elliptic differential operator, this being the operator that is depicted in (IV.1.21) when X is the product cobordism. The operator in the general case is written using slightly different notation in (2.61) of [T3]. This operator has a natural Fredholm incarnation when the respective Y_- version of $\mathfrak{f}_s$ is constant on a neighborhood of $\mathfrak{c}_-$ and the Y_+ version is constant on a neighborhood of $\mathfrak{c}_+$. Use $\imath_{\mathfrak{d}}$ to denote the corresponding Fredholm index. By way of a relevant example, $\imath_{\mathfrak{d}}$ is equal to $\mathfrak{f}_s(\mathfrak{c}_+) - \mathfrak{f}_s(\mathfrak{c}_-)$ when X and the associated data define the product cobordism. Section 8.7 associates an integer, $\imath_{\mathfrak{d}+}$, to $\mathfrak{d}$ which is defined without preconditions on $\mathfrak{c}_-$ and $\mathfrak{c}_+$. The latter is equal to the maximum of $\imath_{\mathfrak{d}}$ and 0 in the case when $\imath_{\mathfrak{d}}$ can be defined.

Proposition 3.7 *Assume that X obeys the conditions in Sections 2.2 and 3.3, that the metric on X obeys (2.8) and (3.12) for a given $L > 100$, and that w_X obeys the conditions in (2.10) and (2.11) for a given $L_* \geq L + 4$. Then there exists $\kappa > \pi$ such that for any given $c \geq \kappa$, there exists κ_c with the following significance: Suppose that the respective pairs of metric and version of w on Y_- and Y_+ define c-tight data. Fix $\mathrm{r} > \kappa_c$ and fix μ_- and μ_+ from the respective Y_- and Y_+ versions of Ω with $\mathcal{P}$-norm less than 1 so as to define (2.9) on X . Let $\mathfrak{d}$ denote an instanton solution to these equations with $\imath_{\mathfrak{d}+} \leq c$. Use $\mathfrak{c}_-$ and $\mathfrak{c}_+$ to denote the respective $s \to -\infty$ and $s \to \infty$ limits of $\mathfrak{d}$. Then $\mathfrak{a}(\mathfrak{c}_-) - \mathfrak{a}(\mathfrak{c}_+) < \mathrm{r}^{2-1/c}$.*

This proposition is proved in Section 8.7.

3.6 The cases when Y_Z is from $\{M \sqcup (S^1 \times S^2), Y\}$, $\{Y_k\}_{k=0,\dots,\mathbf{G}}$, or $\{Y_k \sqcup (S^1 \times S^2)\}_{k=0,\dots,\mathbf{G}-1}$

The body of this article is concerned with $2\mathrm{G}+3$ specific versions of Y_Z, these being as follows: The first manifold of interest is M and $S^1 \times S^2$ and the second is the manifold Y form Section II.1. The next $\mathrm{G}+1$ manifolds are labeled as $\{Y_k\}_{k=0,\dots\mathrm{G}}$ with a given $k \in \{0,\dots,\mathrm{G}\}$ version being the manifold that is obtained from M by attaching the handle $\mathcal{H}_0$ as directed in Part 2 of Section II.1a and attaching k of the handles from the set $\{\mathcal{H}_\mathfrak{p}\}_{\mathfrak{p}\in\Lambda}$ as directed in Part 1 of Section II.1a. Note in this regard that Y and Y_G are the same manifold, but they are distinguished by certain geometric data. The last G manifolds of interest are the disjoint unions of the various $k \in \{0,\dots,\mathrm{G}-1\}$ versions of Y_k and $S^1 \times S^2$.

Part 1: Let Y_Z denote the disjoint union of M and $S^1 \times S^2$. To see about the constraints in Sections 3.2, take Z to be the disjoint union of M and S^3. The handle $\mathcal{H}_0$ is attached to S^3 so as to obtain $S^1 \times S^2$. Write S^1 as $\mathbb{R}/(2\pi\mathbb{Z})$ and let t denote the corresponding affine coordinate. Use the spherical coordinates (θ,ϕ) for S^2. The loop γ is the $\theta = 0$ circle in $S^1 \times S^2$.

To see about w and the metric, consider first their appearance on $S^1 \times S^2$. Take the 2-form w on $S^1 \times S^2$ to be $\sin\theta\, d\theta\, d\phi$ and the metric to be $\mathrm{H}dt^2 + d\theta^2 + \sin^2\theta d\phi^2$ with H denoting a positive constant. If the first Chern class of $\det(\mathbb{S}|_M)$ is torsion, take $w = 0$ on M and take any smooth metric. If the first Chern class of $\det(\mathbb{S}|_M)$ is not torsion, take a metric on M such that the associated harmonic 2-form with de Rham cohomology class that of $c_1(\det(\mathbb{S}|_M))$ has non- degenerate zeros. Take w in this case to be this same harmonic 2-form. By way of a parenthetiral remark, a sufficiently generic metric on M will have this property. See, for example, [Ho] for a proof that such is the case.

The data just described obeys the conditions in Section 3.2. Use Proposition 3.6 to see that this data is also c-tight for a suitably large version of c.

Part 2: Let Y_Z denote the manifold Y that is described in Section II.1. Suffice it to say for now that Y is obtained from Y_0 via a surgery that attaches some positive number of 1-handles to the M_δ part of Y_0. This number is denoted by G.

The 2-form w is described in Section II.1e. See also Part 3 of Section IV.1a. Let b_1 denote the first Betti number of M. Part 2 of Section II.1d describes a set of $b_1 + 1$ closed integral curves of the kernel of w that have geometric intersection number 1 with each u = constant 2-sphere in $\mathcal{H}_0$. One of these curves intersects $\mathcal{H}_0$ as the $\theta = 0$ arc. This is the curve $\gamma^{(z_0)}$ in the notation from Part 2 of Section III.1a. Use the latter for γ. It follows from what is said in (II.1.5) and Part 2 of Section II.1d that the γ has a tubular neighborhood with coordinates as described in Section 3.2 such that the 2-form w has the desired appearance. Section II.1e and (IV.1.5) describe a closed 1-form on Y that can be used to satisfy the requirements in the third bullet in (3.5). This 1-form is denoted by $\upsilon_\diamond$.

A set of Riemannian metrics on Y are described in Part 5 of Section IV.1a that have the desired form on $\mathcal{H}_0$. Although not stated explicitly, a metric of the sort that is described in Part 5 of Section IV.1a can be chosen so that it has the desired behavior on some small radius tubular neighborhood of γ. Note that the set of metrics under consideration are obtained from the choice of an almost complex structure on the kernel of a 1-form $\hat{a}$ given in (IV.1.6). These almost complex structures are taken from the set $\mathcal{J}_{ech}$ that is described in Theorem II.A.1 and Section III.1c. None of the conclusions in [KLT2]-[KLT4] are compromised if the almost complex structure from $\mathcal{J}_{ech}$ is chosen near γ so that the metric obeys the constraints in (3.6). To be sure, the chosen almost complex structure must have certain genericity properties to invoke the propositions and theorems in these papers. These genericity results are used to preclude the existence of certain pseudoholomorphic subvarieties in $\mathbb{R} \times Y$. An almost complex structure giving a metric near γ that obeys (3.6) is not generic. Even so, the subvarieties that must be excluded can be excluded using a suitably almost generic almost complex structure from the subset described in $\mathcal{J}_{ech}$ that give a metric that is described by (3.6) near γ. What follows is the key observation that is used to prove this: The curves to be excluded have image via the projection from $\mathbb{R} \times Y$ that intersects the complement of small radius neighborhoods of γ. A detailed argument for the existence of the desired almost complex structures from $\mathcal{J}_{ech}$ amounts to a relatively straightforward application of the Sard-Smale theorem along the lines used in the proof of Theorem 4.1 in [HT2].

It follows from Lemma IV.2.5 and Proposition IV.2.7 that the metric just described together with w define a c-tight data set on Y for a suitably large choice for c.

Part 3: This part of the subsection considers the case when Y_Z is some $k \in \{0, \dots, \mathrm{G}\}$ version of Y_k. As noted previously, the manifold Y_k is obtained from M by attaching the handle $\mathcal{H}_0$ in the manner that is described in Part 2 of Section II.1a and attaching k

of the handles from the set $\{\mathcal{H}_{\mathfrak{p}}\}_{\mathfrak{p}\in\Lambda}$ as described in Part 1 of Section II.1a. Part 3 in Section II.1a defines a subset $M_\delta \subset M$ and the constructions of both Y and Y_k identify M_δ as a subset of both. The curve $\gamma^{(z_0)}$ that was introduced above in Part 2 sits in the latter part of Y and so it can be viewed using this identification as a curve in Y_k. Use this Y_k incarnation of $\gamma^{(z_0)}$ for the curve γ.

The proposition that follows says what is needed with regards to the 2-form w and the metric to use on Y_k.

Proposition 3.8 *Fix $k \in \{0, \cdots, \mathrm{G}\}$. There exists a nonempty set of Riemannian metrics on Y_k with the following two properties: Let w denote the metric's associated harmonic 2-form with de Rham cohomology class that of $c_1(\det(\mathbb{S}))$. Then w has non-degenerate zeros. Moreover, the metric and w obey the conditions in Section 3.2.*

This proposition is proved in Section 9.2.

The set of metrics in Proposition 3.8 is denoted by *Met* in what follows. Take the metric on Y_k from this set and take w to be the associated harmonic 2-form with de Rham cohomology class that of $c_1(\det(\mathbb{S}))$. Proposition 3.6 asserts that the resulting data set is c-tight for a suitably large choice of c.

Part 4: This part of the subsection discusses the case when Y_Z is the disjoint union some $k \in 0, \cdots, \mathrm{G}-1$ version of Y_k and $S^1 \times S^2$. The metric on Y_0 comes from the Proposition 3.8's set *Met* , and the 2-form w on Y_k is the corresponding harmonic 2-form with de Rham cohomology class that of $c_1(\det(\mathbb{S}))$. Any smooth metric can be chosen for a given $S^1 \times S^2$ component. The class $c_1(\det(\mathbb{S}))$ is taken equal to zero on each $S^1 \times S^2$ component and this understood, the 2-form w is identically zero on each such component.

What is said in Proposition 3.6 implies that the resulting data set is c-tight for a suitably large choice of c.

3.7 Cobordisms with Y_+ and Y_- either Y, $M \sqcup (S^1 \times S^2)$, from $\{Y_k\}_k$, or $\{Y_k \sqcup (S^1 \times S^2)\}_k$

The first proposition concerns the product cobordisms when Y_Z is one of the manifolds from the set Y, $M \sqcup (S^1 \times S^2)$, $\{Y_k\}_{k=0,\dots,\mathrm{G}}$ or $\{Y_k \sqcup (S^1 \times S^2)\}_{k=0,\dots,\mathrm{G}-1}$. The subsequent propositions concern certain cobordisms of the sort described in Section 3.3 with Y_+ and Y_- as follows:

- One is Y and the other is Y_{G}.
- One is Y_k and the other is $Y_{k-1} \sqcup (S^1 \times S^2)$ for some $k \in \{1, ..., \mathrm{G}\}$.
- One is Y_0 and the other is $M \sqcup (S^1 \times S^2)$.

These propositions assume implicitly that the metric and version of w on these manifolds are those supplied by the relevant part of Section 3.6. In particular, the metric and w on $M \sqcup (S^1 \times S^2)$ is described by Part 1 of Section 3.6, and this data on Y is described in Part 2 of Section 3.6. Meanwhile, the metric on the relevant $k \in \{0, \ldots, \mathrm{G}\}$ version of Y_k is from the set *Met* and w is the associated harmonic 2-form with de Rham cohomology class that of $c_1(\det(\mathbb{S}))$.

Proposition 3.9 *Let Y_Z denote either $M \sqcup (S^1 \times S^2)$ or Y or some $k \in \{0, \ldots, \mathrm{G}\}$ version of Y_k with the 2-form w and metric as described in the preceding paragraph. Given $\imath \geq 0$, there exists $\kappa > \pi$ with the following significance: Fix any $\mathrm{r} > \kappa$ and a 1-form $\mu \in \Omega$ with either $\mathcal{P}$-norm less than $e^{-\mathrm{r}^2}$ or $\mathcal{P}$-norm less than 1 but vanishing on $\mathcal{H}_0 \cup U_\gamma$. Use this data with the metric and w to define the product cobordism $X = \mathbb{R} \times Y_Z$ as prescribed in Section 2.2. Suppose that $\mathfrak{c}_-$ and $\mathfrak{c}_+$ are solutions to the (r, μ)-version of (2.4) on Y_Z with $|\mathfrak{f}_s(\mathfrak{c}_+) - \mathfrak{f}_s(\mathfrak{c}_-)| \leq \imath$; and suppose that $\mathfrak{d}$ is an instanton solution to (2.9) on X with $s \to -\infty$ limit equal to $\mathfrak{c}_-$ and $s \to \infty$ limit equal to $\mathfrak{c}_+$. Then $\mathrm{X}(\mathfrak{c}_+) \geq \mathrm{X}(\mathfrak{c}_-)$.*

Proof. This follows directly from Propositions 3.3 and 3.6 given what is said in Section 3.6 about w and the metric. □

The next proposition describes cobordisms between Y_0 and $M \sqcup (S^1 \times S^2)$ of the sort that obey the conditions in Section 3.3.

Proposition 3.10 *Take the metric on $M \sqcup (S^1 \times S^2)$ and harmonic 2-form w to be as described in Part 1 of Section 3.6. The metric on $M \sqcup (S^1 \times S^2)$ determines a corresponding set of metrics in the Y_0 version of Met. Choose a metric from this set and take w on Y_0 to be the associated harmonic 2-form with de Rham class $c_1(\det(\mathbb{S}))$. Denote one of Y_0 or $M \sqcup (S^1 \times S^2)$ by Y_- and the other by Y_+. There exists a cobordism that obeys the conditions in Section 3.3 and the conditions in the list below. This list uses X to denote the cobordism manifold.*

- *The function s on X has exactly one critical point. This critical point has index 3 when $Y_- = Y_0$ and index 1 when $Y_- = M \sqcup (S^1 \times S^2)$.*
- *There is a metric on X with an associated self-dual 2-form that are (c, r)-compatible if L and c are sufficiently large and if $r > \pi$.*

This proposition is proved in Section 9.4.

The next proposition uses C to denote the cylinder in Proposition 3.10's cobordism that is described by the first bullet of (3.9). The proposition also reintroduces the notation in (3.15).

Proposition 3.11 *Take w and the metric on Y_0 and on $M \sqcup (S^1 \times S^2)$ to be as described in Proposition 3.10. Denote one of Y_0 or $M \sqcup (S^1 \times S^2)$ by Y_- and the other by Y_+. Take the cobordism space X, the metric on X, and the associated closed 2-form w_X to be as described by Proposition 3.10. Given $k \geq 0$, there exists $\kappa > \pi$ with the following significance: Fix $\mathrm{r} > \kappa$ and 1-forms μ_- and μ_+ from the Y_-- and Y_+-versions of Ω with either $\mathcal{P}$-norm less than $e^{-\mathrm{r}^2}$ or with $\mathcal{P}$-norm less than 1 but vanishing on the Y_-- and Y_+-versions of $\mathcal{H}_0 \cup U_\gamma$. Let $\mathfrak{d} = (A, \psi)$ denote an instanton solution to the resulting version of (2.9) with $\imath_{\mathfrak{d}+} \leq k$. Then $i \int_C F_{\hat{A}} \geq 0$.*

Proof. The proposition follows directly from Propositions 3.4, 3.7 and 3.10 given what is said in Section 3.6 about the respective Y_0 and $M \sqcup (S^1 \times S^2)$ metrics and versions of w. □

The next set of propositions are analogs of Propositions 3.10 and 3.11 in the case when one of Y_- and Y_+ is some $k \in \{1, \cdots, \mathrm{G}\}$ version of Y_k and the other is $Y_{k-1} \sqcup (S^1 \times S^2)$, or when one is Y and the other is Y_G. The propositions that follow assume that $c_1(\det \mathbb{S})$ on each $k \in \{0, \ldots, \mathrm{G}\}$ version of Y_k vanishes on the cross-sectional spheres in any $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_\mathfrak{p}$ and that it has pairing 2 with the cross-sectional spheres in $\mathcal{H}_0$. This class is also assumed to be zero on the $S^1 \times S^2$ component of any $k \in \{0, \ldots, \mathrm{G}\}$ version of $Y_{k-1} \sqcup (S^1 \times S^2)$. Meanwhile, its restriction to the $H_2(M; \mathbb{Z})$ summand from the associated Mayer-Vietoris sequence for the various $k \in \{0, \ldots, \mathrm{G}\}$ versions of $H_2(Y_k; \mathbb{Z})$ is assumed to be independent of k.

Proposition 3.12 *There exists, for each $k \in \{0, \ldots, \mathrm{G}\}$, a subset to be denoted by $\mathrm{Met}(Y_k)$ in the Y_k version of Met with the following significance: Let $\mathrm{Met}(Y_0)$ denote the subset from Proposition 3.10. For each $k \in \{1, \ldots, \mathrm{G}\}$, take a metric from an open subset of $\mathrm{Met}(Y_{k-1})$ and a metric on $S^1 \times S^2$ to define a metric on $Y_{k-1} \sqcup (S^1 \times S^2)$. Take w on $Y_{k-1} \sqcup (S^1 \times S^2)$ to be the associated harmonic 2-form with de Rham class $c_1(\det \mathbb{S})$. The chosen metric determines a corresponding subset of metrics $\mathrm{Met}(Y_k) \subset$ Met. Take a metric from the latter subset and take w to be the associated harmonic 2-form with de Rham class $c_1(\det(\mathbb{S}))$. Take Y_- to be one of Y_k and $Y_{k-1} \sqcup (S^1 \times S^2)$, and take Y_+ to be the other. There exists a cobordism that obeys the conditions in Section 3.2 and the conditions listed below. This list uses X to denote the cobordism manifold.*

- *The function s on X has precisely 1 critical point. This critical point has index 3 when Y_+ has the $S^1 \times S^2$ component and it has index 1 when Y_- has the $S^1 \times S^2$ component.*
- *There is a metric on X with an associated self-dual 2-form that are (c, r)-compatible if L, c and $r > \pi$.*

This proposition is proved in Section 9.5.

The next proposition considers the case when one of Y_- and Y_+ is Y and the other is Y_{G}.

Proposition 3.13 *Take w and the metric on Y to be as described in the opening paragraphs of this subsection. Take the metric on Y_{G} from a certain non-empty subset of $\mathrm{Met}(Y_{\mathrm{G}})$ and take w on Y_{G} to be the associated harmonic 2-form with de Rham class that of $c_1(\det(\mathbb{S}))$. Take Y_- to be one of Y and Y_{G} and take Y_+ to be the other. There exists a cobordism that obeys the conditions in Section 3.2 and the conditions listed below. This list uses X to denote the cobordism manifold.*

- *The function s on X has no critical points.*
- *There is a metric on X with an associated self-dual 2-form that are (c, r)-compatible if L, c and $r > \pi$.*

The proof of Proposition 3.13 is in Section 9.6.

The upcoming proposition uses C to denote the cylinder in Proposition 3.12 and 3.13's cobordism that is described by the first bullet of (3.9). Notation from (3.15) is also used.

Proposition 3.14 *Let X denote one of the cobordism manifolds that are described in Propositions 3.12 and 3.13 with $c_1(\det(\mathbb{S}))$ and the 2-form and metrics on Y_-, Y_+ and X as described therein. Given $\imath \geq 0$, there exists $\kappa > \pi$ with the following property: Fix $\mathrm{r} > \kappa$ and 1-forms μ_- and μ_+ from the Y_-- and Y_+-versions of Ω with either $\mathcal{P}$-norm less than $e^{-\mathrm{r}^2}$ or with $\mathcal{P}$-norm less than 1 but vanishing on the Y_- and Y_+ versions of $\mathcal{H}_0 \cup U_\gamma$. Let $\mathfrak{d} = (A, \psi)$ denote an instanton solution to the resulting version of (2.9) with $\imath_{\mathfrak{d}+} \leq \imath$. Then $i\int_C F_{\hat{A}} \geq 0$.*

Proof. The proposition follows directly from Propositions 3.4, 3.7, 3.12 and 3.13 given what is said in Section 3.6. □

3.8 Filtered Floer homologies and filtration-preserving chain maps

This subsection is divided into two parts. In the first part, we associate to each triple (Y_Z, w, γ) described in Section 3.2 a system of filtered monopole Floer homologies $HM^\circ(Y_Z, \mathrm{r}w; \Lambda_\gamma)$, for $\circ = -, \infty, +, \wedge$ and $\mathrm{r} > \pi$, in the manner described in Section 3.1. Recall the constraint on the cohomology class $[w]$ from Part 3 of Section 3.2. Together with the first bullet of (3.5), this implies that $CM_*(Y_Z, \mathrm{r}w; \Lambda_\gamma)$ is associated with a negative-monotone, non-balanced perturbation. For reasons that will become clear momentarily, we use $CM^\circ(Y_Z, \langle w\rangle; \Lambda_\gamma)$ to denote $CM_*(Y_Z, \mathrm{r}w; \Lambda_\gamma)$ for $\mathrm{r} \gg \pi$ and similarly for its homology. (The notation $\langle w\rangle$ stands for the ray $\mathbb{R}^+[w] \subset H^2(Y_Z; \mathbb{R})$. This includes, as special cases, the triple $(\underline{M}, \underline{w}, \underline{\gamma})$ in [L] ($\underline{M}$ is denoted by Y_0 in this article), and the triple $(Y, w, \gamma^{(z_0)})$ in Section II.1a.

In the second part, a filtration-preserving chain map from $CM^-(Y_-, \langle w_-\rangle; \Lambda_{\gamma_-})$ to $CM^-(Y_+, \langle w_+\rangle; \Lambda_{\gamma_+})$ is associated to each triple (X, ϖ_X, C) described in Section 3.3. To explain the notation, X is a cobordism from the 3-manifold Y_- to Y_+, ϖ_X is a self-dual 2-form on X related to w_- and w_+ as prescribed by (2.10). What is denoted by C signifies an embedded surface in X, with ends $\gamma_- \subset Y_-$ and $\gamma_+ \subset Y_+$; see the second bullet of (3.9).

Part 1: To accomplish this task, begin by introducing the (partially defined) integral 1-cocycle on $\mathcal{B}^\sigma(Y_Z)$ defining Λ_γ. This local system associates each $\mathfrak{c} \in \mathcal{Z}_{w,\mathrm{r}}$ the group algebra $\mathbb{K}[\mathbb{Z}] = \mathbb{K}[U, U^{-1}]$. To each $\mathfrak{d} \in \mathcal{M}_1(\mathfrak{c}_-, \mathfrak{c}_+)$ it associates $U^{n(\mathfrak{d})} \in \mathrm{End}(\mathbb{K}[U, U^{-1}])$, where $n(\mathfrak{d}) = \mathrm{x}(\mathfrak{c}_+) - \mathrm{x}(\mathfrak{c}_-)$. Here, x is the "modified holonomy function" given in (3.8). Lemma 3.1 asserts that $n(\mathfrak{d}) \in \mathbb{Z}$ for $\mathfrak{c}_-, \mathfrak{c}_+ \in \mathcal{Z}_{w,\mathrm{r}}$. Following the recipe in Section 3.1, we then set $(CM^\infty, \partial^\infty)$ to be the monopole Floer complex with twisted coefficients:

$$\begin{aligned} CM^\infty &= \mathbb{K}[U, U^{-1}](\mathcal{Z}_{w,\mathrm{r}}), \\ \partial^\infty \mathfrak{c}_- &= \sum_{\mathfrak{c}_+ \in \mathcal{Z}_{w,\mathrm{r}}} \sum_{\mathfrak{d} \in \mathcal{M}_1(\mathfrak{c}_-, \mathfrak{c}_+)/\mathbb{R}} \mathrm{sign}(\mathfrak{d})\, U^{n(\mathfrak{d})} \mathfrak{c}_+ \quad \text{for } \mathfrak{c}_- \in \mathcal{Z}_{w,\mathrm{r}}. \end{aligned}$$

The monotonicity condition guarantees that the sum here is finite. The $\mathrm{sign}(\mathfrak{d}) \in \{\pm 1\}$ in the preceding expression is assigned according to the orientation convention laid out in [KM].

One may regard CM^∞ as a chain complex over $\mathbb{K}$, generated by $\hat{\mathcal{Z}}_{w,\mathrm{r}} = \mathcal{Z}_{w,\mathrm{r}} \times \mathbb{Z}$. The generating set $\hat{\mathcal{Z}}_{w,\mathrm{r}}$ lies in $\tilde{\mathcal{B}}^\sigma = \mathcal{C}^\sigma/\mathcal{G}_\gamma$, a $\mathbb{Z}$-covering of $\mathcal{B}^\sigma$. Here, $\mathcal{G}_\gamma \subset \mathcal{G}$ consists of smooth maps $u\colon Y_Z \to S^1$, with $\deg(u|_\gamma) = 0$. Multiplication by U^n then corresponds to a deck-transformation on this $\mathbb{Z}$-covering, and the condition on

$c_1(\det \mathbb{S})$ set forth in Part 3 of Section 3.2 then implies that $\deg U = -2$. The grading set of $\hat{\mathcal{Z}}_{w,\mathrm{r}}$ is an affine space over $\mathbb{Z}/c_Z\mathbb{Z}$, where $c_Z \in 2\mathbb{Z}$ is the gcd of the values of $c_1(\det \mathbb{S})$ on $H_2(Z;\mathbb{Z})$ according the splitting (3.4).

Remark 3.15 (a) Here, we use the same notation U for the map on monopole Floer complexes described in Part 2 of Section 2.6 and deck transformation here. This is because for the kind of Y_Z considered in this article, they turns out identical by the arguments for the last bullet of Proposition IV.7.6.

(b) The way grading on a monopole Floer complex with local coefficients follow the way they are defined in some literature, e.g. what is called a Floer-Novikov complex [L1]. The book [KM] does not seem to contain an explicit discussion on the grading of Floer complex with local coefficients.

Suppose that (Y_Z, w) define c-tight data for $c > 1$ (cf. Definition 3.5). Take X to be the product cobordism $\mathbb{R} \times Y_Z$, $w_X = w + ds \wedge *w$, and $C = \mathbb{R} \times \gamma \subset X$. Let $\mathfrak{d} \in \mathcal{M}(\mathfrak{c}_-, \mathfrak{c}_+)$ be as in Proposition 3.4. In this case, $i\int_C F_{\hat{A}} = 2\pi(\mathrm{X}(\mathfrak{c}_+) - \mathrm{X}(\mathfrak{c}_-))$ and Proposition 3.4 asserts that one has $n(\mathfrak{d}) \geq 0$. Thus,

$$CM^- = \mathbb{K}(\mathcal{Z} \times \mathbb{Z}^{\geq 0}) \subset CM^\infty$$

is a subcomplex of $\mathbb{K}[\mathbb{Z}^{\geq 0}] = \mathbb{K}[U]$-modules. One may then introduce

$$CM^+ = CM^\infty \,/\, CM^-, \quad \widehat{CM} = CM^- \,/ U\, CM^- .$$

The resulting short exact sequences

$$\begin{aligned} &0 \to CM^- \to CM^\infty \to CM^+ \to 0 \quad \text{and} \\ &0 \to U\,CM^- \to CM^- \to \widehat{CM} \to 0 \end{aligned} \tag{3.16}$$

induce the *fundamental exact sequences* on the homologies. As the $\bigwedge^* H_1(Y_Z)/\mathrm{Tors}$-action on the monopole Floer complexes commute with U, the exact sequences above preserve the $\mathbf{A}_\dagger$-module structure.

In Section 3.6, the assumption that (Y_Z, w) is c-tight is verified for the particular

manifolds listed therein. In particular,

(3.17)

- when $Y_Z = Y$ and its assciated $\mathfrak{s}$, w, γ and metric are as in Part 2 of Section 3.6,
$$HM^\circ(Y, \langle w\rangle; \Lambda_\gamma) = \mathrm{H}^\circ(Y) = \mathrm{H}^\circ_{SW}$$
in the notation of [KLT1, KLT4].
- when $Y_Z = Y_k$, $k \in \{0, \cdots, \mathrm{G}\}$ and its associated $\mathfrak{s}$, w, γ and metric are in Part 3 of Section 3.6, the corresponding $HM^\circ(Y, \langle w\rangle; \Lambda_\gamma)$ are intrumental in the proof for Theorem 1.1. Recalling that Y_0 and its assciated $\mathfrak{s}$, w, γ and metric are respectively what was denoted by $\underline{M}$ $\underline{\mathfrak{s}}$, $\underline{w}$, $\underline{\gamma}$ in [L], we observe that
$$HM^\circ(Y_0, \langle w\rangle; \Lambda_\gamma) = \mathrm{HMT}^\circ$$
introduced in [L].

Note that $CM^\circ(Y_Z, \langle w\rangle; \Lambda_\gamma)$, $HM^\circ(Y_Z, \langle w\rangle; \Lambda_\gamma)$ introduced above implicitly depend on r and $(\mathfrak{T}, \mathfrak{S})$. According to the convention set forth in Section 1.3, this is permissible if there are chain homotopies between the monopole Floer complexes associated with different parameters preserving the $\mathbf{A}_\dagger$-module structure. This is justified by combining the arguments proving Proposition IV.1.4 with what is said in the upcoming Part 2.

Part 2: We now consider chain maps induced by (non-product) cobordisms X described in Section 3.3. To begin, we introduce an X-morphism from Λ_{γ_-} to Λ_{γ_+}. (See Definition 23.3.1 in [KM] for "X-morphism"). This is done in a way similar to the definition of Γ_C in Equation (23.8) in [KM]. In [KM], a "cobordism" from Y_- to Y_+ refers to a compact 4-manifold with boundary $Y_+ \sqcup (-Y_-)$. This corresponds to the compact part of our X, denoted by $X_c = s^{-1}([-L_{tor}, L_{tor}])$. The surface $C \cap X_c$ plays the role of the singular 2-chain ν in (23.8) of [KM]. It has boundary $\gamma_+ - \gamma_-$, with $\gamma_+ \simeq \gamma \simeq \gamma_-$. Given $\mathfrak{c}_- \in \mathcal{Z}_{w_-,\mathrm{r}}(Y_-)$ and $\mathfrak{c}_+ \in \mathcal{Z}_{w_+,\mathrm{r}}(Y_+)$, let $\mathfrak{d}$ denote an element in $\mathcal{B}^\sigma(X)$ with $s \to -\infty$ limit $\mathfrak{c}_-$ and $s \to \infty$ limit $\mathfrak{c}_+$. Then Γ_C is an isomorphism from $\Gamma_{\gamma_-}(\mathfrak{c}_-) \simeq \mathbb{R}$ to $\Gamma_{\gamma_+}(\mathfrak{c}_+) \simeq \mathbb{R}$ given by multiplication by $e^{\frac{i}{2\pi}\int_C F_{\mathbb{A}}}$. The analog of Γ_C in our setting, denoted Λ_C below, is given by an homomorphism from $\Lambda_{\gamma_-}(\mathfrak{c}_-) \simeq \mathbb{K}[U, U^{-1}]$ to $\Lambda_{\gamma_+}(\mathfrak{c}_+) \simeq \mathbb{K}[U, U^{-1}]$ for each pair $\mathfrak{c}_-$, $\mathfrak{c}_+$. This is given by multiplication with $U^{n(\mathfrak{d})}$, where

$$n(\mathfrak{d}) = \frac{i}{2\pi}\int_C F_{\hat{A}} = \mathrm{x}_{\gamma_+}(\mathfrak{c}_+) - \mathrm{x}_{\gamma_-}(\mathfrak{c}_-), \tag{3.18}$$

the right most equality being a consequence of the Stokes' theorem. This is again an integer according to Lemma 3.1. With Λ_C in place, given $u \in C^k(\mathcal{U}; \mathbb{K})$ in the notation

of Section 2.6, we define the map

$$m^{\infty}[u](X,\langle w_X\rangle;\Lambda_C)\colon\ CM^{\infty}(Y_-)=\mathbb{K}[U,U^{-1}](\mathcal{Z}_{w_-,\mathrm{r}})\to CM^{\infty}(Y_+)=\mathbb{K}[U,U^{-1}](\mathcal{Z}_{w_+,\mathrm{r}})$$

by the following rule:

$$\mathcal{Z}_{w_-,\mathrm{r}}(Y_-)\ni\mathfrak{c}_-\mapsto\sum_{\mathfrak{c}_+\in\mathcal{Z}_{w_+,\mathrm{r}}(Y_+)}\sum_i\langle u,\mathcal{M}_k(X,\mathfrak{c}_-,\mathfrak{c}_+)\rangle\,U^{n(\mathfrak{d}_i)}\,\mathfrak{c}_+,$$

where i runs through each connected component of $\mathcal{M}_k(X,\mathfrak{c}_-,\mathfrak{c}_+)$; and for every i, $\mathfrak{d}_i$ is an element in the corresponding connected component. In order for the sum on the right hand side to be well defined, we assume that $H^2(X,Y_-)=0$ and w_X satisfies (2.16).

To proceed, suppose (Y_-,w_-), (Y_+,w_+) are c-tight and consider $C(X,\langle w_X\rangle;\Lambda_C)|_{CM^-(Y_-)}$. Suppose furthermore that (X,w_X) satisfies the conditions in Propositions 3.4 and 3.7. By these propositions, the integers $n(\mathfrak{d}_i)$ in (3.18) are non-negative, implying that the image of $C(X,\langle w_X\rangle;\Lambda_C)|_{CM^-(Y_-)}$ under m^{∞} lies in $CM^-(Y_+)$. Use

$$m^-[u](X,\langle w_X\rangle;\Lambda_C)\colon\ CM^-(Y_-)\to CM^-(Y_+)$$

to denote this map. It is straightforward to see that both m^{∞} and m^- are chain maps, given that CM^{∞} is a variant of monopole Floer complexes, and the non-negativity of the integers $n(\mathfrak{d})$ appearing in the formulae for ∂^{∞} and m^{∞}. These then induce homomorphisms between the respective homologies:

$$HM_*(X,\langle w_X\rangle;\Lambda_C)\colon\ HM^{\circ}(Y_-,\langle w_-\rangle;\Lambda_{\gamma_-})\to HM^{\circ}(Y_+,\langle w_+\rangle;\Lambda_{\gamma_+})$$

for $\circ=-,\infty$. Like those in Part 1, these maps preserve the $\mathbf{A}_\dagger$-module structure.

4 Some homological algebra

As mentioned in Section 1, the purpose of this section is to review the algebraic background for the upcoming Proposition 5.9. The latter is used to relate the fomula for monopole Floer homology of a connected sum, given in Proposition 6.4 below, in terms of the monopole Floer homology with balanced perturbation that appears in Theorem 1.1 and Theorem 1.4. This computation turns out to be a simplest manifestation of the so-called "Koszul duality", well-known in certain circles. For a sampling of literature on this subject, see e.g. [C], [Ko], [GKM]. The variant most relevant to this article is discussed in [GKM], which relates the ordinary chain complex of an S^1-space, equipped with an $H_*(S^1)$-module structure capturing the S^1-action, with the S^1-equivariant chain complexes of the same space, which are naturally endowed

with $H^*(BS^1)$-module structures. We need however only a small portion of the full machinery in [GKM]. Thus, in this section we give a self-contained though elementary exposition of the relevant part of this story, tailored to our needs.

4.1 Terminology and conventions

By a *modules over* $H^*(BS^1)$ we mean a chain complex with a module structure over $\mathbb{K}[u]$, where u acts as a chain map of degree -2. The prime examples of such modules in this article are the monopole Floer complexes. In parallel, a *module over* $H_*(S^1)$ stands for a chain complex with a module structure over $\mathbb{K}[y]$, where y acts as a degree 1 chain map. An example that appears later is the chain complex to compute the monopole Floer homology of a connected sum, see (6.1) in Proposition 6.2. Meanwhile, a graded homology module H_* will be viewed as a chain complex with zero differentials. In this article, we assume that the grading group of these modules to be such that it has a reduction to an absolute $\mathbb{Z}/2\mathbb{Z}$-grading. In this manner, the notion of even and odd chain maps are well-defined, as well as the notion of graded commutativity. We use capital letters U, Y to denote the chain maps corresponding to the action of u, y.

Definition 4.1 A *morphism* from one module over $H^*(BS^1)$ to another is a $\mathbb{K}$-chain map which commutes with U-actions. Morphisms between $H_*(S^1)$-modules are defined similarly, with Y replaced by U. We shall also often encounter a weaker notion: a *p-morphism* between two $H^*(BS^1)$-modules is a $\mathbb{K}$-chain map which commutes with U-actions up to $\mathbb{K}$-chain homotopy.

4.2 From $H^*(BS^1)$-modules to $H_*(S^1)$-modules

Given a module (C, ∂_C) over $H^*(BS^1)$, we define the module $S_U(C)$ over $H_*(S^1) = \mathbb{K}[y]$ as follows:

$$\big(S_U(C), S_U(\partial_C)\big) = (C \otimes \mathbb{K}[y],\ \partial_C \otimes \jmath + U \otimes y), \tag{4.1}$$

where the homomorphism $\jmath\colon \mathbb{K}[y] \to \mathbb{K}[y]$ is defined by

$$\jmath(a + by) = a - by \quad \text{for } a, b \in \mathbb{K},$$

and the y-action is simply the multiplication $1 \otimes y$. ($\jmath$ was denoted by σ in [L]. Cp. Equation (5.1) therein).

To see that $S_U(C)$ is indeed a chain complex, note that the condition $S_U(\partial_C)^2 = 0$ is equivalent to the pair of identities $\partial_C^2 = 0$, and $[\partial_C, U] = 0$.

Lemma 4.2 *A p-morphism Φ between two $H^*(BS^1)$-modules $(C_{(1)}, \partial_{(1)})$, $(C_{(2)}, \partial_{(2)})$ induces an $H_*(S^1)$-module morphism $S_U(\Phi)$ between $S_{U_{(1)}}(C_{(1)})$ and $S_{U_{(2)}}(C_{(2)})$, where $U_{(1)}$, $U_{(2)}$ denote the u-action on $C_{(1)}$, $C_{(2)}$ respectively. Furthermore,*

- *$S_U(\Phi)$ is injective if Φ is injective, and it is surjective if Φ is surjective;*
- *Let Φ' be another p-morphism of $H^*(BS^1)$-modules from $(C_{(1)}, \partial_{(1)})$ to $(C_{(2)}, \partial_{(2)})$. Then $\Phi + \Phi'$ is a p-morphism as well, and*
$$S_U(\Phi + \Phi') = S_U(\Phi) + S_U(\Phi').$$
- *Let Ψ be a p-morphism of $H^*(BS^1)$-modules from $(C_{(2)}, \partial_{(2)})$ to $(C_{(3)}, \partial_{(3)})$. Then $\Psi \circ \Phi$ is a p-morphism as well, and*
$$S_U(\Psi \circ \Phi) = S_U(\Psi) \circ S_U(\Phi). \tag{4.2}$$

Proof. As a p-morphism, Φ satisfies both

$$\Phi\, \partial_{(1)} - (-1)^{\deg(\Phi)} \partial_{(2)}\, \Phi = 0 \qquad \text{and}$$
$$\Phi\, U_{(1)} - U_{(2)}\, \Phi = K_\Phi\, \partial_{(1)} + (-1)^{\deg(\Phi)} \partial_{(2)}\, K_\Phi,$$

for a $\mathbb{K}$-linear homomorphism K_Φ. This is equivalent to the identity

$$S_U(\Phi)\, S_U(\partial_{(1)}) - (-1)^{\deg(\Phi)} S_U(\partial_{(2)})\, S_U(\Phi) = 0, \tag{4.3}$$

where $S_U(\Phi)\colon\ C_{(1)} \otimes \mathbb{K}[y] \to C_{(2)} \otimes \mathbb{K}[y]$ is defined as

$$S_U(\Phi) = \Phi \otimes \jmath^{\deg(\Phi)} + K_\Phi \otimes y.$$

This verifies that $S_U(\Phi)$ is a chain map. Moreover, since the y-action on $S_U(C_{(1)})$, $S_U(C_{(2)})$ is multiplication by $1 \otimes y$, it is immediate that $S_U(\Phi)$ commutes with the y-actions on both sides. The claim that S_U preserves injectivity and surjectivity can be checked directly from the definition of $S_U(\Phi)$.

Since the construction of $S_U(\Phi)$ is linear, the second item in the statement of the Lemma is obvious.

To verify the third bullet about the composition of p-morphisms, let

$$S_U(\Psi) = \Psi \otimes \jmath^{\deg(\Psi)} + K_\Psi \otimes y. \tag{4.4}$$

Then (4.2) is straightforward to verify, given that

$$K_{\Psi\circ\Phi} = K_\Psi \circ \Phi + (-1)^{\deg(\Psi)} \Psi \circ K_\Phi.$$

The fact that $\Psi \circ \Phi$ is a p-morphism follows directly from (4.3) and its analog for Ψ.

The first bullet may be directly verified after writing out the definition of $S_U(\Phi)$ explicitly. More is said in the proof of Lemma 4.6 below. □

Definition 4.3 Two $H^*(BS^1)$-modules $(C_{(1)}, \partial_{(1)})$, $(C_{(2)}, \partial_{(2)})$ are said to be *p-homotopic* if there exist p-morphisms $\Phi\colon C_{(1)} \to C_{(2)}$, $\Psi\colon C_{(2)} \to C_{(1)}$, and $H_1\colon C_{(1)} \to C_{(1)}$, $H_2\colon C_{(2)} \to C_{(2)}$, such that

$$\Psi \circ \Phi - \mathrm{Id}_{(1)} = [\partial_{(1)}, H_1], \quad \Phi \circ \Psi - \mathrm{Id}_{(2)} = [\partial_{(2)}, H_2].$$

They are said to be *homotopic* if Φ, Ψ, H_1, H_2 are morphisms. The notion of two $H_*(S^1)$-modules being *homotopic* is defined similarly.

Lemma 4.4 *Suppose two $H^*(BS^1)$-modules $(C_{(1)}, \partial_{(1)})$, $(C_{(2)}, \partial_{(2)})$ are* p-homotopic *via p-morphisms $\Phi\colon C_{(1)} \to C_{(2)}$, $\Psi\colon C_{(2)} \to C_{(1)}$ as above. Then the $H_*(S^1)$-modules $S_U(C_{(1)})$, $S_U(C_{(2)})$ are homotopic via the maps $S_U(\Phi)$ and $S_U(\Psi)$.*

Proof. By assumption, there exist H_1, H_2 such that Φ, Ψ satisfy:

$$\Psi \circ \Phi - \mathrm{Id}_{(1)} = [\partial_{(1)}, H_1], \quad \Phi \circ \Psi - \mathrm{Id}_{(2)} = [\partial_{(2)}, H_2].$$

We need to verify the identities:

$$\begin{aligned} S_U(\Psi) \circ S_U(\Phi) - \mathrm{Id}_{(1)} &= [S_U(\partial_{(1)}), S_U(H_1)];\\ S_U(\Phi) \circ S_U(\Psi) - \mathrm{Id}_{(2)} &= [S_U(\partial_{(2)}), S_U(H_2)];\\ [S_U(\Phi), Y] &= 0;\\ [S_U(\Psi), Y] &= 0. \end{aligned}$$

It suffices to verify the first and the third identities, since the second and the fourth are entirely parallel.

To verify the first identity, use (4.2) and the fact that $\Psi \circ \Phi - \mathrm{Id}_{(1)} = [\partial_{(1)}, H_1]$ to reduce it to

$$S_U(\mathrm{Id}_{(1)}) = \mathrm{Id}\,.$$

This holds by taking $\Psi = \mathrm{Id}_{(1)}$ and $K_\Psi = 0$ in (4.4).

To verify the third identity, simply plug in the definition of $S_U(\Phi)$ and $Y = I \otimes y$. □

4.3 From $H_*(S^1)$-modules to $H^*(BS^1)$-modules

First, introduce the $\mathbb{K}[u]$-modules:

$$\begin{aligned} V^- &:= u\mathbb{K}[u],\\ V^\infty &:= \mathbb{K}[u, u^{-1}],\\ V^+ &:= \mathbb{K}[u, u^{-1}]/u\mathbb{K}[u],\\ V^\wedge &:= \mathbb{K}[u]/u\mathbb{K}[u]. \end{aligned} \tag{4.5}$$

These modules by definition fit into the short exact sequences:

$$0 \to V^- \overset{i_V}{\hookrightarrow} V^\infty \to V^+ \to 0, \tag{4.6}$$

$$0 \to V^- \xrightarrow{u} V^- \to V^\wedge \to 0. \tag{4.7}$$

We shall frequently view these four modules as a system, and write them collectively as V°. The same convention applies to the various system of modules we construct out of these four below.

Definition 4.5 ([J, L]) Given a module (C, ∂_C) over $H_*(S^1)$, we define the following system of modules over $H^*(BS^1) = \mathbb{K}[u]$:

$$\big(E_Y^\circ(C), E_Y(\partial_C)\big) := (C \otimes V^\circ,\ \partial_C \otimes 1 + Y \otimes u) \quad \text{for } \circ = -, \infty, +, \wedge, \tag{4.8}$$

where the u-action is the multiplication $1 \otimes u$.

The fact that $E_Y(\partial_C)^2 = 0$ again follows directly from the definition of $H_*(S^1)$-modules: $\partial_C^2 = 0$, $[Y, \partial_C] = 0$, $Y^2 = 0$. By taking tensor product of (4.6), (4.7) with C, one has the following corresponding short exact sequences of $\mathbb{K}[u]$-modules:

$$0 \to E_Y^-(C) \xrightarrow{\mathrm{Id} \otimes i_V} E_Y^\infty(C) \to E_Y^+(C) \to 0, \tag{4.9}$$

$$0 \to E_Y^-(C) \xrightarrow{\mathrm{Id} \otimes u} E_Y^-(C) \to E_Y^\wedge(C) \to 0. \tag{4.10}$$

It is also straightforward to verify that the maps in the above exact sequences commute with $E_Y(\partial_C)$, and therefore they induce the following long exact sequences of $H^*(BS^1)$-modules associated to (C, ∂_C):

$$\cdots \to H_*(E_Y^-(C)) \xrightarrow{i_{V*}} H_*(E_Y^\infty(C)) \to H_*(E_Y^+(C)) \xrightarrow{\delta_{V*}} H_{*-1}(E_Y^-(C)) \to \cdots \tag{4.11}$$

$$\cdots \to H_*(E_Y^-(C)) \xrightarrow{u} H_*(E_Y^-(C)) \to H_*(E_Y^\wedge(C)) \to H_{*-1}(E_Y^-(C)) \to \cdots \tag{4.12}$$

We call (4.11), (4.12) the (resp. first and second) *fundamental exact sequences* for the $H_*(S^1)$-module C. For convenience of later reference, we denote the short exact sequences of $H^*(BS^1)$-modules (4.9), (4.10) by $\mathbb{E}_Y(C)$, $\mathbf{E}_Y(C)$ respectively. Correspondingly, the long exact sequences (4.11), (4.12) are denoted by $H(\mathbb{E}_Y(C))$, $H(\mathbf{E}_Y(C))$. It is straightforward to verify the assertion in the following Lemma and so we leave it to the reader to check that

Lemma 4.6 *A morphism ϕ between two $H_*(S^1)$-modules $(C_{(1)}, \partial_{(1)})$, $(C_{(2)}, \partial_{(2)})$ induces a system of $H^*(BS^1)$-module morphisms*

$$E^\circ(\phi)\colon\ E^\circ_{Y_{(1)}}(C_{(1)}) \to E^\circ_{Y_{(2)}}(C_{(2)})$$
$$\phi \mapsto \phi \circ 1$$

for $\circ = -, \infty, +, \wedge$*, where* $Y_{(1)}$*,* $Y_{(2)}$ *denote the* y*-actions on* $(C_{(1)}, \partial_{(1)})$*,* $(C_{(2)}, \partial_{(2)})$ *respectively. Moreover:*

- $E^\circ(\phi)$ *is injective if* ϕ *is injective; it is surjective if* ϕ *is surjective.*
- *Let* ϕ' *be another morphism of* $H_*(S^1)$*-modules from* $(C_{(1)}, \partial_{(1)})$ *to* $(C_{(2)}, \partial_{(2)})$*. Then* $\phi + \phi'$ *is an* $H_*(S^1)$*-morphism as well, and*
$$E^\circ(\phi + \phi') = E^\circ(\phi) + E^\circ(\phi').$$
- *Let* ψ *be another morphism of* $H_*(S^1)$*-modules from* $(C_{(2)}, \partial_{(2)})$ *to* $(C_{(3)}, \partial_{(3)})$*. Then* $\psi \circ \phi$ *is an* $H_*(S^1)$*-morphism as well, and*
$$E^\circ(\psi \circ \phi) = E^\circ(\psi) \circ E^\circ(\phi). \tag{4.13}$$
- *The system of morphisms* $E^\circ(\phi)$ *combine to define morphisms of short exact sequences of* $H^*(BS^1)$*-modules*
$$\mathbb{E}(\phi)\colon \mathbb{E}_Y(C_{(1)}) \to \mathbb{E}_Y(C_{(2)}) \quad \textit{and} \quad \mathbf{E}(\phi)\colon \mathbf{E}_Y(C_{(1)}) \to \mathbf{E}_Y(C_{(2)}).$$
Correspondingly, their induced maps on homologies $H_*(E_Y(\phi))$ *combine to define morphisms of long exact sequences of* $H^*(BS^1)$*-modules*
$$H(\mathbb{E}_Y(\phi))\colon H(\mathbb{E}_Y(C_{(1)})) \to H(\mathbb{E}_Y(C_{(2)})) \quad \textit{and}$$
$$H(\mathbf{E}_Y(\phi))\colon H(\mathbf{E}_Y(C_{(1)})) \to H(\mathbf{E}_Y(C_{(2)})).$$

Proof. The proofs are straightforward; thus we shall say no more than making the following remarks: Both E_Y° and S_U preserve injectivity and surjectivity due to the same reason, namely they can be written in polynomial form (in u and y respectively, which defines a filtration), where their 0-th order term takes the form of a tensor product of the original morphism and an automorphism. This in turn implies that both of them takes short exact sequences to short exact sequences. □

Lemma 4.7 *Let* $C_{(1)}$*,* $C_{(2)}$ *denote homotopic* $H_*(S^1)$*-modules. Then* $E_Y^\circ(C_{(1)})$*,* $E_Y^\circ(C_{(2)})$ *are homotopic* $H^*(BS^1)$*-modules.*

Proof. By assumption, there exist morphisms $\Phi\colon C_{(1)} \to C_{(2)}$, $\Psi\colon C_{(2)} \to C_{(1)}$ and $H_1\colon C_{(1)} \to C_{(1)}$, $H_2\colon C_{(2)} \to C_{(2)}$ such that
$$\Psi \circ \Phi - \mathrm{Id}_{(1)} = [\partial_{(1)}, H_1], \quad \Phi \circ \Psi - \mathrm{Id}_{(2)} = [\partial_{(2)}, H_2]. \tag{4.14}$$
Lemma 4.6 claims $E^\circ(\Phi)\colon E_Y^\circ(C_{(1)}) \to E_Y^\circ(C_{(2)})$, $E^\circ(\Psi)\colon E_Y^\circ(C_{(2)}) \to E_Y^\circ(C_{(1)})$ are systems of morphisms. Meanwhile, the desired identities are:
$$E^\circ(\Psi) \circ E^\circ(\Phi) - \mathrm{Id}_{(1)} = [E_Y(\partial_{(1)}), E^\circ(H_1)], \quad E^\circ(\Phi) \circ E^\circ(\Psi) - \mathrm{Id}_{(2)} = [E_Y(\partial_{(2)}), E^\circ(H_2)]. \tag{4.15}$$

We shall only verify the first identity, since the second is similar. For this purpose, apply E° to the first identity in (4.14), then apply Lemma 4.6 and subtract the first line of (4.15) to the resulting identiy. This leads to

$$E^\circ(\mathrm{Id}_{(1)}) - \mathrm{Id} = [Y, H_1] \otimes u.$$

This is true because of the definition of E° and the fact that H_1 is a morphism. $\square$

4.4 Koszul duality

The functors S_U and E^- may be viewed as inverses of each other in the following sense.

Proposition 4.8 **(a)** *Let (C, ∂_C) be an $H^*(BS^1)$-module. Then there is a system of isomorphisms of $H^*(BS^1)$-modules:*

$$H_*(E_Y^\circ S_U(C)) \simeq H_*(C \otimes_{\mathbb{K}[u]} V^\circ). \tag{4.16}$$

Moreover, these isomorphisms have the following naturality properties:

(i) they are natural with respect to p-morphisms of $H^*(BS^1)$-modules, and

(ii) they combine to define isomorphisms of long exact sequences of $H^*(BS^1)$-modules

$$H(\mathbb{E}_Y S_U(C)) \simeq H(C \otimes_{\mathbb{K}[u]} \mathbb{V}), \quad \text{and} \quad H(\mathbf{E}_Y S_U(C)) \simeq H(C \otimes_{\mathbb{K}[u]} \mathbf{V}).$$

Here, $H(C \otimes_{\mathbb{K}[u]} \mathbb{V})$ and $H(C \otimes_{\mathbb{K}[u]} \mathbf{V})$ respectively denote the long exact sequence induced by the short exact sequences of $H^*(BS^1)$-modules:

$$0 \to C \otimes_{\mathbb{K}[u]} V^- \to C \otimes_{\mathbb{K}[u]} V^\infty \to C \otimes_{\mathbb{K}[u]} V^+ \to 0 \quad \text{and}$$
$$0 \to C \otimes_{\mathbb{K}[u]} V^- \xrightarrow{1 \otimes u} C \otimes_{\mathbb{K}[u]} V^- \to C \otimes_{\mathbb{K}[u]} V^\wedge \to 0.$$

(b) *Let (C, ∂_C) be an $H_*(S^1)$-module. Then there is an isomorphism of $H_*(S^1)$-modules:*

$$H_*(S_U E_Y^-(C)) \simeq H_*(C).$$

Proof. Part (a): Written out explicitly,

$$E_Y^\circ S_U(C) = C \otimes \mathbb{K}[y] \otimes V^\circ,$$
$$E_Y S_U(\partial_C) = \partial_C \otimes \jmath \otimes 1 + U \otimes y \otimes 1 + 1 \otimes y \otimes u.$$

View this as a filtered complex by the total degree in the $C\otimes V^\circ$ factor. Then the E_1-term of the associated spectral sequence is simply

$$C\otimes\mathbb{K}\{y\}\otimes V^\circ/\big((U\otimes y\otimes 1+1\otimes y\otimes u)(C\otimes\mathbb{K}\{1\}\otimes V^\circ)\big)\ \simeq C\otimes_{\mathbb{K}[u]}V^\circ, \tag{4.17}$$

with differential d_1 given by $-\partial_C$. This spectral sequence degenerates at E_2, and we have $H_*(E^\circ_Y S_U(C))\simeq H_*(C\otimes_{\mathbb{K}[u]}V^\circ)$ as claimed. As the u-action on $E^\circ_Y S_U(C)$ is $1\otimes 1\otimes u$ and the u-action on C is U, the quotient in (4.17) shows that the isomorphism preserves the $\mathbb{K}[u]$-module structure. Property (ii) also follows immediately from this computation.

On the other hand, given a p-morphism Φ between two $H^*(BS^1)$-modules $(C_{(1)},\partial_{(1)})$, $(C_{(2)},\partial_{(2)})$, by Lemmas 4.2 and 4.6, there is a corresponding system of morphisms of $H^*(BS^1)$-modules $E^\circ_Y S_U(\Phi)$. The naturality property (i) follows from the fact that these morphisms preserve the filtration.

Part (b): Written out explicitly,

$$S_U E^-_Y(C)=C\otimes u\mathbb{K}[u]\otimes\mathbb{K}[y],$$
$$S_U E^-_Y(\partial_C)=\partial_C\otimes 1\otimes\sigma+Y\otimes u\otimes\sigma+1\otimes u\otimes y.$$

Filtrate by the total degree of the factor $C\otimes u\mathbb{K}[u]$ as in the previous part. Then the E_1 term is

$$C\otimes u\mathbb{K}[u]\otimes R\{y\}/\big((1\otimes u\otimes y)(C\otimes u\mathbb{K}[u]\otimes R\{1\})\big)\simeq C,$$

on which d_1 acts as $-\partial_C$. The spectral sequence again degenerates at E_2, yielding the claimed isomorphism $H_*(S_U E^-_Y(C))\simeq H_*(C)$. To see that the module structure agree, note that a cycle in the E_1 term given by an element $-z_1\in C$, $\partial_C z_1=0$ corresponds to a cycle in $S_U E^-_Y(C)$ of the form $Z_0\otimes 1+z_1\otimes u\otimes y$, where $Z_0\in C\otimes u\mathbb{K}[u]$ satisfies

$$-((Y\otimes u)Z_0)\otimes 1+((1\otimes u)Z_0)\otimes y-(Yz_1)\otimes u^2\otimes y=0.$$

In other words, $Z_0=-(Yz_1)\otimes u$, and the cycle in $S_U E^-_Y(C)$ has the form $-(Yz_1)\otimes u\otimes 1+z_1\otimes u\otimes y$. The y-action $1\otimes 1\otimes y$ takes this element to $-(Yz_1)\otimes u\otimes y$, while the element corresponding to $-Yz_1\in C$ in the E_1 term is $-(Yz_1)\otimes u\otimes y$ as well, since $Y^2=0$. □

Remark 4.9 (a) Spelled out explicitly, (4.16) says that $H_*(E^-_Y S_U(C))\simeq H_*(C)$, and $H_*(E^\infty_Y S_U(C))$ is the localization of $H_*(C)$ as a $\mathbb{K}[u]$-module. On the other hand, note that since $V^\wedge=\mathbb{K}[u]/u\mathbb{K}[u]$, $E^\wedge S_U(\partial_C)$ reduces to $S_U(\partial_C)\otimes 1$, and therefore $H_*(E^\wedge_Y S_U(C))\simeq H_*(S_U(C))$.

(b) The constructions E^-_Y, E^∞_Y, E^+_Y above are directly copied from J. Jones's formulation of the "co-Borel", "Tate", and Borel (the usual) versions of equivariant homologies [J]. It is proved in [GKM] that S_U, and E_Y induce isomorphisms of derived categories.

5 Balanced Floer homologies from monotone Floer chain complexes

This section re-introduces the fourth flavor of monopole Floer homology denoted by HM^{tot} in [L], now renamed $\widetilde{HM}$ in deference to Donaldson's notation. (Cf. p.187 of [D]) This definition is a natural by-product of a re-interpretation of $\mathring{HM}_*(M,\mathfrak{s},c_b)$ in terms *purely of the* $\mathbb{K}[U]$*-module* $\hat{C}_*(M,\mathfrak{s},c_b)$ (Corollary 5.3 in [L], re-stated as Proposition 5.9 below). This result enables us to appeal to the third author's "$SW = G$r" program, which in our context was carried out in part IV of this series [KLT4]. The latter constructed an isomorphism from an appropriate variant of *ech* to a *negative monotone* version of monopole Floer homology, which is in turn related to the balanced version via the following theorem of Kronheimer and Mrowka:

Theorem 5.1 ([KM] Theorem 31.5.1) *Let* $\widehat{C}_*(M,\mathfrak{s},c_b)$ *and* $C_*(M,\mathfrak{s},c_-) = \widehat{C}_*(M,\mathfrak{s},c_-)$ *respectively denote the Seiberg-Witten-Floer chain complexes with balanced and negative monotone perturbations. Then there is a chain homotopy equivalence from the former to the latter. In particular,* $\widehat{HM}_*(M,\mathfrak{s},c_b) \simeq HM_*(M,\mathfrak{s},c_-)$.

To be more precise, the statement of Theorem 31.5.1 in [KM] concerns only the Floer homologies. However, the chain homotopy equivalence referred to above was constructed in its proof.

Remark 5.2 The variant of *ech* relevant in this series of papers is related to the negative monotone version of monopole Floer homology, and therefore to $\widehat{HM}_*$ by the preceding theorem of [KM]. This is because the stable Hamiltonian structure used to define the relevant *ech* is associated to an *nonexact* closed 2-form. Note in contrast that the ordinary embedded contact homology associated to a contact structure is related to $\widetilde{HM}_*$ instead, since the relevant 2-form in this case is exact. As such, it belongs to the positive monotone situation, and the companion theorem to the one just cited states that $\widetilde{HM}_*(M,\mathfrak{s},c_b) \simeq HM_*(M,\mathfrak{s},c_+)$.

5.1 Some properties of the maps i, j, p

In this section, unless otherwise specified, $\mathring{C}_* = \mathring{C}_*(M,c_b)$ denotes the monopole Floer chain complex associated to an oriented Riemannian, Spin^c 3-manifold with a *balanced perturbation*. Similarly, let $\mathring{HM}_* = \mathring{HM}_*(M,c_b)$.

Recall from Proposition 22.2.1 in [KM] that $\widehat{HM}_*$, $\widecheck{HM}_*$, $\overline{HM}_*$ are related by a long exact sequence

(5.1) $$\cdots \to \overline{HM}_* \xrightarrow{i_*} \widecheck{HM}_* \xrightarrow{j_*} \widehat{HM}_* \xrightarrow{p_*} \overline{HM}_{*-1} \xrightarrow{i_*} \cdots,$$

which we shall call the *fundamental exact sequence of monopole Floer homologies*. The maps i_*, j_*, p_* in the sequence above are respectively induced by maps:

$$i\colon \bar{C} \to \check{C}, \quad j\colon \check{C} \to \hat{C}, \quad p\colon \hat{C} \to \bar{C},$$

which, written in block form with respect to the decomposition

(5.2) $$\bar{C} = C^s \oplus C^u, \quad \check{C} = C^o \oplus C^s, \quad \hat{C} = C^o \oplus C^u$$

are given by

(5.3) $$i = \begin{bmatrix} 0 & -\partial_o^u \\ 1 & -\partial_s^u \end{bmatrix}, \quad j = \begin{bmatrix} 1 & 0 \\ 0 & -\bar{\partial}_u^s \end{bmatrix}, \quad p = \begin{bmatrix} \partial_s^o & \partial_s^u \\ 0 & 1 \end{bmatrix}.$$

It is shown in [KM] that they are respectively chain maps of degree 0, degree 0, degree -1.

Lemma 5.3 *The maps i, j, p are p-isomorphisms of $H^*(BS^1)$-modules.*

Proof. It is verified in [KM] that $[\mathring{\partial}, \mathring{U}] = 0$ for the to-, from, and bar-versions of monopole Floer chain complexes. A straightforward though tedious computation using (5.3) shows that:

(5.4) $$\begin{aligned} i\bar{U} - \check{U}i + K_i\bar{\partial} + \check{\partial}K_i &= 0, \\ j\check{U} - \hat{U}j + K_j\check{\partial} + \hat{\partial}K_j &= 0, \\ p\hat{U} - \bar{U}p - K_p\hat{\partial} + \bar{\partial}K_p &= 0, \end{aligned}$$

where K_i, K_j, K_p, written in block form with respect to the same decompositions (5.2), are:

$$K_i = \begin{bmatrix} 0 & -U_o^u \\ 0 & -U_s^u \end{bmatrix}, \quad K_j = \begin{bmatrix} 0 & 0 \\ 0 & -\bar{U}_u^s \end{bmatrix}, \quad K_p = \begin{bmatrix} U_s^o & U_s^u \\ 0 & 0 \end{bmatrix}.$$

□

As was explained in the proof of Lemma 4.2, the identities (5.4) can be rewritten as

$$\begin{aligned} S_U(i)S_U(\bar{\partial}) - S_U(\check{\partial})S_U(i) &= 0, \\ S_U(j)S_U(\check{\partial}) - S_U(\hat{\partial})S_U(j) &= 0, \\ S_U(p)S_U(\hat{\partial}) + S_U(\bar{\partial})S_U(p) &= 0, \end{aligned}$$

where $S_U(i)$, $S_U(j)$, $S_U(p)$, when written in block form with respect to the same decomposition (5.2), as matrices with coefficients in $\mathbb{K}[y]$, are given as follows:

$$S_U(i) = \begin{bmatrix} 0 & -n_o^u \\ 1 & -n_s^u \end{bmatrix}, \quad S_U(j) = \begin{bmatrix} 1 & 0 \\ 0 & -\bar{n}_u^s \end{bmatrix}, \quad S_U(p) = \begin{bmatrix} N_s^o & N_s^u \\ 0 & 1 \end{bmatrix},$$

where

$$\begin{aligned} \bar{n}_u^s &= \bar{\partial}_u^s + \bar{U}_u^s y, \quad & n_*^* &= \partial_*^* + U_*^* y, \\ \bar{N}_u^s &= \bar{\partial}_u^s \jmath + \bar{U}_u^s y, \quad & N_*^* &= \partial_*^* \jmath + U_*^* y, \end{aligned}$$

these being homomorphisms of $\mathbb{K}[y]$-modules for any pair of super- and subscripts $*$ among u, s, o.

Lemma 5.4 *The induced maps from i, j, p fit into the following long exact sequences:*

(5.5)
$$\cdots \longrightarrow H_*(S_U(\bar{C})) \xrightarrow{S_U(i)_*} H_*(S_U(\check{C})) \xrightarrow{S_U(j)_*} H_*(S_U(\hat{C})) \xrightarrow{S_U(p)_*} H_{*-1}(S_U(\bar{C})) \xrightarrow{S_U(i)_*} \cdots;$$

(5.6)
$$\begin{aligned} \cdots \longrightarrow H_*(E_Y^\circ S_U(\bar{C})) &\xrightarrow{E_Y^\circ S_U(i)_*} H_*(E_Y^\circ S_U(\check{C})) \xrightarrow{E_Y^\circ S_U(j)_*} H_*(E_Y^\circ S_U(\hat{C})) \\ &\xrightarrow{E_Y^\circ S_U(p)_*} H_{*-1}(E_Y^\circ S_U(\bar{C})) \xrightarrow{E_Y^\circ S_U(i)_*} H_{*-1}(E_Y^\circ S_U(\check{C})) \cdots. \end{aligned}$$

The first sequence is a sequence of $H_*(S^1)$-modules, and the second one is a sequence of $H^*(BS^1)$-modules.

Proof. *Part (a).* The proof is based on a modification of the proof of Proposition 22.2.1 in [KM]. Recall from [KM] the definition of a "mapping cone of $-p$" $(\check{E}, \check{e})$:

$$\check{E} = \hat{C} \oplus \bar{C}, \quad \check{e} = \begin{bmatrix} \hat{\partial} & 0 \\ p & \bar{\partial} \end{bmatrix}.$$

The short exact sequence associated with $(\check{E}, \check{e})$, $0 \to \bar{C} \to \check{E} \to \hat{C} \to 0$, induces a long exact sequence connecting the triple $\overline{HM}$, $H(\check{E})$, $\widehat{HM}$, with connecting map p_*. [KM] shows that $\check{E}$ is chain-homotopic to $\check{C}$. The following diagram summarizes the construction:

$$\begin{tikzcd} & & \check{C} \arrow[d, "k"] & & \\ 0 \arrow[r] & \bar{C} \arrow[ur, "i"] \arrow[r, "\bar{\imath}"] & \check{E} \arrow[r, "\bar{\jmath}"] \arrow[d, "l"] & \hat{C} \arrow[ul, "j"'] \arrow[r] & 0 \\ & & \check{C} & & \end{tikzcd}$$

where
$$k = \begin{bmatrix} j \\ \Pi_s \end{bmatrix}, \quad l = \begin{bmatrix} \Pi_o & i \end{bmatrix},$$
with
$$\Pi_s \colon \check{C} = C^o \oplus C^s \to \overline{C} = C^s \oplus C^u,$$
$$\Pi_o \colon \hat{C} = C^o \oplus C^u \to \check{C} = C^o \oplus C^s,$$
$$\Pi_u \colon \overline{C} = C^s \oplus C^u \to \hat{C} = C^o \oplus C^u$$
denoting projections to the s, o, u components respectively.

In terms of these, the proof in [KM] reduces to the verification of the following identities:

$$lk = \mathrm{Id}, \tag{5.7}$$
$$kl = \mathrm{Id} + \check{e}K + K\check{e}, \tag{5.8}$$
$$j = \bar{j}k, \tag{5.9}$$
$$ki - \bar{i} = \check{e}(K\bar{i}) + (K\bar{i})\,\bar{\partial}, \tag{5.10}$$

where
$$K = \begin{bmatrix} 0 & -\Pi_u \\ 0 & 0 \end{bmatrix}.$$

We now want to apply the preceeding constructions and identities to the S_U-versions. To do so, first observe that the identities (5.4) imply that $\check{E}$ is an $H^*(BS^1)$-module, with the U-map given by
$$U_{\check{e}} = \begin{bmatrix} \hat{U} & 0 \\ K_p & \bar{U} \end{bmatrix}.$$

With this defined, it is straightforward to check that $\bar{i}$, $\bar{j}$ are $H^*(BS^1)$-morphisms. We can then use what was said in the previous subsection to form the $H_*(S^1)$-modules and morphisms $(S_U(\check{E}), S_U(\check{e}))$, $S_U(\bar{i})$, $S_U(\bar{j})$. Lemma 4.2 ensures that

$$0 \to S_U(\overline{C}) \xrightarrow{S_U(\bar{i})} S_U(\check{E}) \xrightarrow{S_U(\bar{j})} S_U(\hat{C}) \to 0$$

is a short exact sequence of $H_*(S^1)$-modules. Meanwhile, the identities (5.4) can be used again to verify that:

$$k\check{U} - U_{\check{E}}k + \check{K}_j\check{\partial} + \check{e}\check{K}_j = 0,$$
$$lU_{\check{E}} - \check{U}l + \check{K}_i\check{e} + \check{\partial}\check{K}_i = 0,$$

where
$$\check{K}_j = \begin{bmatrix} -K_j \\ 0 \end{bmatrix}, \quad \text{and} \quad \check{K}_i = \begin{bmatrix} 0 & K_i \end{bmatrix}.$$

This means that l, k are both p-morphisms of $H^*(BS^1)$-modules. By Lemma 4.2 we can then form the $H_*(S^1)$-module morphisms $S_U(l)$, $S_U(k)$. The analogs of (5.7), (5.8)

$$S_U(l)\,S_U(k) = \mathrm{Id}, \quad S_U(j) = S_U(\bar{j})\,S_U(k) \tag{5.11}$$

now follow readily from the naturality property of S_U described in Lemma 4.2. Meanwhile, the analogs of (5.8), (5.10)

$$\begin{aligned} S_U(k)S_U(l) &= \mathrm{Id} + S_U(\check{e})(K\otimes \jmath) + (K\otimes \jmath)\,S_U(\check{e}),\\ S_U(k)S_U(i) - S_U(\bar{\imath}) &= S_U(\check{e})\mathbb{K} + \mathbb{K}S_U(\bar{\partial}), \quad \mathbb{K} := (K\otimes \jmath)S_U(\bar{\imath}) \end{aligned} \tag{5.12}$$

reduce to the following identities:

$$\begin{aligned} &K_j\Pi_o - \Pi_u K_p = 0\\ &\Pi_s K_i + K_p \Pi_u = 0\\ &\hat{U}\,\Pi_u - \Pi_u\bar{U} - K_j\, i + jK_i = 0; \end{aligned}$$

and these can be directly verified. This proves (5.5).

To verify (5.6), we simply apply E_Y° to the S_U-version of [KM]'s constructions and identities obtained above. Since we have shown that $(S_U(\check{E}), S_U(\check{e}))$, $S_U(\bar{\imath})$, $S_U(\bar{j})$, $S_U(l)$, $S_U(k)$ are $H_*(S^1)$-morphisms, Lemma 4.6 implies that $(E_Y^\circ S_U(\check{E}), E_Y^\circ S_U(\check{e}))$, $E_Y^\circ S_U(\bar{\imath})$, $E_Y^\circ S_U(\bar{j})$, $E_Y^\circ S_U(l)$, $E_Y^\circ S_U(k)$ are $H^*(BS^1)$-morphisms, and the analogs of the identities (5.11), (5.12) follow without much ado by applying E_Y° to them and the naturality properties of E_Y° described in Lemma 4.6. □

5.2 The $\bar{C}_*$ complex and localization

Lemma 5.5 $H_*(S_U(\bar{C})) = 0$.

Proof. To compute $H_*(S_U(\bar{C}))$, write

$$S_U(\bar{C}) = \bar{C}\otimes \mathbb{K}[y], \quad S_U(\bar{\partial}) = \bar{\partial}\otimes \jmath + \bar{U}\otimes y.$$

Filtrate this complex by the degree in the factor $\bar{C}$; this is done just as in the proof of Proposition 4.8. The E_1-term is $\overline{HM}_*$; and d_1 is the u-map on $\overline{HM}_*$. We claim that this map is invertible, and therefore $H_*(S_U(\bar{C}))$ vanishes.

To see that this is indeed the case, write

$$\bar{C} = C_{\mathbb{T}}\otimes \mathbb{K}[x, x^{-1}], \tag{5.13}$$

where $C_{\mathbb{T}}$ is the Morse complex of a Morse function on the torus of flat connections, which is finitely generated. Recall that a generator $a\otimes x^m$ for $C_{\mathbb{T}}\otimes \mathbb{K}[x, x^{-1}]$ corresponds to the m-th eigenvalue of D_a, the Dirac operator with the flat connection,

where the eigenvalues are ordered by their value in $\mathbb{R}$, and $1 = x^0$ corresponds to the minimal positive eigenvalue.

The index of the $C_{\mathbb{T}}$ factor defines a finite length filtration on $\bar{C}$, with respect to which $\bar{U}$ can be written as $\sum_{k=0}^{N} \bar{U}_k$ for some $N \in \mathbb{Z}^{\geq 0}$. However, $\bar{U}_0 = x$ (understood as multiplication), because the only possible contribution to $\bar{U}_0$ comes from the moduli space of instantons from $a \otimes x^m$ to $a \otimes x^{m-1}$; and this consists of the space of gradient flows of the quadratic function $\sum_{m\in\mathbb{Z}} \lambda_m |\xi_m|^2$ on $\mathbb{P}(\mathrm{Span}_{\mathbb{C}}\{\eta_m\}_m)$. Here, η_m denotes a chosen unit-norm eigenvector of λ_m. This moduli space is $\mathbb{C}P^1$. The fact that $\bar{U}_0 = x$ is an invertible operator on $C_{\mathbb{T}} \otimes \mathbb{K}[x, x^{-1}]$ then means that $\bar{U}$ is invertible as well. □

It follows from the preceding Lemma and Lemma 5.4 that $S_U(j)$ induces an $H_*(S^1)$-module isomorphism from $H_*(S_U(\check{C}))$ to $H_*(S_U(\hat{C}))$.

Definition 5.6 (Cf. [L] Equation (5.6)) We call the following group the "total"-version of monopole Floer homology:

$$\widetilde{HM}_* := H_*(S_U(\hat{C})) \simeq H_*(S_U(\check{C})).$$

The motivation for this definition comes from the theory of S^1-equivariant theroy; it is related to the equivariant versions of Floer homologies $\widehat{HM}$, $\widecheck{HM}$, $\overline{HM}$ by properties expected of the homology of their corresponding S^1-space. (The choice of the accent $\sim$ in the notation reflects the fact that this is supposed to come from the space of framed configurations, in accordance with the notation (5.1.1) in [DK]). In particular, the following lemma is a consequence of Proposition 4.8 (a) (ii) and Remarks 4.9 that

Lemma 5.7 *$\widetilde{HM}_*$ is related to $\widehat{HM}_*$ by the following long exact sequence:*

$$\cdots \to \widehat{HM}_* \xrightarrow{U} \widehat{HM}_{*-2} \to \widetilde{HM}_* \to \widehat{HM}_{*-1} \to \cdots . \tag{5.14}$$

The following lemma is invoked in the next subsection:

Lemma 5.8 (Localization) *Let $\hat{C}$, $\bar{C}$, $\widehat{HM}$, $\overline{HM}$ denote the monopole Floer complexes or homologies for a balanced perturbation. Then:*

(a) *The map $i_{V_*}\colon H_*(E_Y^- S_U(\bar{C})) \to H_*(E_Y^\infty S_U(\bar{C}))$ is an isomorphism.*

(b) *The map p_* induces an isomorphism of $\mathbb{K}[u, u^{-1}]$-modules:*

$$p_*\colon \widehat{HM}_* \otimes_{\mathbb{K}[u]} \mathbb{K}[u, u^{-1}] \to \overline{HM}_* \otimes_{\mathbb{K}[u]} \mathbb{K}[u, u^{-1}].$$

Proof. *Part (a).* By Proposition 4.8, it is equivalent to consider the localization map $H_*(\overline{C}) \to H_*(\overline{C} \otimes_{\mathbb{K}[u]} \mathbb{K}[u,u^{-1}])$. However, we saw in the proof of Lemma 5.5 that the u-action is invertible on $H_*(\overline{C})$.

Part (b). $\mathrm{Tor}(\mathbb{K}[u], \mathbb{K}[u,u^{-1}]) = 0$; so we can work at the chain level.

$$H_*(\hat{C} \otimes_{\mathbb{K}[u]} \mathbb{K}[u,u^{-1}]) = H_*((\hat{U}^N \hat{C}) \otimes_{\mathbb{K}[u]} \mathbb{K}[u,u^{-1}])$$

for any $N \in \mathbb{Z}^{\geq 0}$. There are finitely many irreducible Seiberg-Witten solutions; and with a balanced perturbation, the Seiberg-Witten actional functional is real-valued. We can therefore order these finitely many irreducibles by their values of action functional. A nonconstant Seiberg-Witten instanton always decreases the actions unless it is reducible; so for sufficiently large N, $\hat{U}^N \hat{C} \subset C^u$.

Meanwhile, we saw in (5.13) that $C^u = C_{\mathbb{T}} \otimes (x\mathbb{K}[x])$ and $C^s = C_{\mathbb{T}} \otimes \mathbb{K}[x^{-1}]$. We also saw in the proof of Lemma 5.5 that $\overline{U}_0 = x$. Therefore, C^u generates $\overline{C} \otimes_{\mathbb{K}[u]} \mathbb{K}[u,u^{-1}]$. This understood, the assertion follows because we can restrict our attention to C^u and the $u - u$ component of p is the identity. □

5.3 Monopole Floer homologies from twisted tensor products

The modules $S_U(C)$ and $E_Y(C)$ are "twisted tensor products" (in the sense of e.g. [TT], [Lef]), on which $H^*(BS^1)$ and $H_*(S^1)$ respectively act by simple multiplications. On the other hand, the duality theorem Proposition 4.8 tells us the following: On the homological level, we can replace any $H^*(BS^1)$ or $H_*(S^1)$-modules by such twisted tensor products by applying $E_Y^- S_U$ or $S_U E_Y^-$ respectively. We shall reformulate the monopole Floer homologies $\widehat{HM}_*$, $\widecheck{HM}_*$, $\overline{HM}_*$ defined in [KM] accordingly. In addition to these three flavors of monopole Floer homologies, we will introduce a fourth flavor $\widetilde{HM}_*$ from this point of view. These four flavors of monopole Floer homologies will be regarded as a system and denoted collectively as $\mathring{HM}_*$ below. We call $\widehat{HM}_*$, $\widecheck{HM}_*$, $\overline{HM}_*$, $\widetilde{HM}_*$ the *from-, to, bar-, total*-versions of monopole Floer homology respectively. Just as $\widehat{HM}_*$, $\widecheck{HM}_*$, $\overline{HM}_*$ are to be viewed as versions of equivariant homologies of the equivariant Seiberg-Witten-Floer stable homotopy type (represented by a pointed S^1-space) $\mathrm{SWF}(Y, \mathfrak{c})$ that is introduced in [M], what we denoted by $\widetilde{HM}_*$ can be viewed as the (non-equivariant) homology of $\mathrm{SWF}(Y, \mathfrak{c})$ itself.

We now state the main result of this subsection.

Proposition 5.9 *([L] Corollary 5.3) Let $\hat{C}$ denote $\hat{C}(M,\mathfrak{s},c_b)$ and $\mathring{HM}$ denote $\mathring{HM}(M,\mathfrak{s},c_b)$. There is a system of isomorphisms (as $H^*(BS^1)$-modules) from $H_*(E^\circ_Y S_U(\hat{C}))$ to $\mathring{HM}_*$, taking the fundamental exact sequence of equivariant homologies for $S_U(\hat{C})$ to the fundamental exact sequence of monopole Floer homologies. In particular, we have the following commutative diagram of $H^*(BS^1)$-modules:*

(5.15)

$$\begin{tikzcd}
\cdots H_*(E^-_Y S_U(\hat{C})) \arrow[r, "i_{V*}"] \arrow[d] & H_*(E^\infty_Y S_U(\hat{C})) \arrow[r] \arrow[d] & H_*(E^+_Y S_U(\hat{C})) \arrow[r, "\delta_{V*}"] \arrow[d] & H_{*-1}(E^-_Y S_U(\hat{C}))\cdots \arrow[d] \\
\cdots \widehat{HM}_* \arrow[r, "p_*"] & \overline{HM}_{*-1} \arrow[r, "i_*"] & \widetilde{HM}_{*-1} \arrow[r, "j_*"] & \widehat{HM}_{*-1}\cdots
\end{tikzcd}$$

where the vertical arrows are $H^*(BS^1)$-module-isomorphisms.

Proof of Proposition 5.9. Consider the following diagram:

(5.16)

$$\begin{tikzcd}
{} \arrow[d, "E_YS_U(j)"] & {} \arrow[d, "E_YS_U(j)"] & {} \arrow[d, "E_YS_U(j)"] & {} \arrow[d, "E_YS_U(j)"] \\
\cdots H_{*+2}(E^+_Y S_U(\hat{C})) \arrow[r] \arrow[d, "E_YS_U(p)"] & H_{*+1}(E^-_Y S_U(\hat{C})) \arrow[r, "i_{V*}"] \arrow[d, "E_YS_U(p)"] & H_{*+1}(E^\infty_Y S_U(\hat{C})) \arrow[r] \arrow[d, "E_YS_U(p)", "\simeq"'] \arrow[dl, dotted, "\bar{h}"'] & H_{*+1}(E^+_Y S_U(\hat{C}))\cdots \arrow[d, "E_YS_U(p)"] \\
\cdots H_{*+1}(E^+_Y S_U(\bar{C})) \arrow[r] \arrow[d, "E_YS_U(i)"] & H_*(E^-_Y S_U(\bar{C})) \arrow[r, "i_{V*}", "\simeq"'] \arrow[d, "E_YS_U(i)"] & H_*(E^\infty_Y S_U(\bar{C})) \arrow[r] \arrow[d, "E_YS_U(i)"] & H_*(E^+_Y S_U(\bar{C}))\cdots \arrow[d, "E_YS_U(i)"] \\
\cdots H_{*+1}(E^+_Y S_U(\check{C})) \arrow[r, "\delta_{V*}", "\simeq"'] \arrow[d, "E_YS_U(j)", "\simeq"'] & H_*(E^-_Y S_U(\check{C})) \arrow[r, "i_{V*}"] \arrow[d, "E_YS_U(j)"] & H_*(E^\infty_Y S_U(\check{C})) \arrow[r] \arrow[d, "E_YS_U(j)"] & H_*(E^+_Y S_U(\check{C}))\cdots \arrow[d, "E_YS_U(j)", "\simeq"'] \\
\cdots H_{*+1}(E^+_Y S_U(\hat{C})) \arrow[r] \arrow[d, "E_YS_U(p)"] \arrow[ur, dotted, "\check{h}"'] & H_*(E^-_Y S_U(\hat{C})) \arrow[r, "i_{V*}"] \arrow[d, "E_YS_U(p)"] & H_*(E^\infty_Y S_U(\hat{C})) \arrow[r] \arrow[d, "E_YS_U(p)", "\simeq"'] & H_*(E^+_Y S_U(\hat{C}))\cdots \arrow[d, "E_YS_U(p)"] \\
\cdots H_{*+1}(E^+_Y S_U(\bar{C})) \arrow[r] \arrow[d, "E_YS_U(i)"] & H_*(E^-_Y S_U(\bar{C})) \arrow[r, "i_{V*}", "\simeq"'] \arrow[d, "E_YS_U(i)"] & H_*(E^\infty_Y S_U(\bar{C})) \arrow[r] \arrow[d, "E_YS_U(i)"] & H_*(E^+_Y S_U(\bar{C}))\cdots \arrow[d, "E_YS_U(i)"] \\
{} & {} & {} & {}
\end{tikzcd}$$

All rows and columns above are exact sequences of $H^*(BS^1)$-modules: the rows are fundamental exact sequences of equivariant homologies of $S_U(\hat{C})$, $S_U(\bar{C})$, $S_U(\check{C})$, and the columns are the exact sequences from (5.6).

By Proposition 4.8, the exact sequence in the second column is isomorphic to the first fundamental exact sequence of the monopole Floer homologies (5.1), namely the second row in (5.15). Therefore we shall henceforth replace the second column by

$$\cdots \xrightarrow{j_*} \widehat{HM}_* \xrightarrow{p_*} \overline{HM}_* \xrightarrow{i_*} \widecheck{HM}_* \xrightarrow{j_*} \widehat{HM}_{*-1} \xrightarrow{p_*} \cdots .$$

Our goal is therefore to construct an isomorphism from the exact sequence in the first or fourth row to the exact sequence in the second column.

$$\begin{tikzcd}
\cdots H_{*+1}(E_Y^- S_U(\hat C)) \arrow[r, "i_{V*}"] \arrow[d, "\hat{}"'] & H_{*+1}(E_Y^\infty S_U(\hat C)) \arrow[r] \arrow[d, "\bar h"'] & H_{*+1}(E_Y^+ S_U(\hat C)) \cdots \arrow[d, "\check h"'] \\
\cdots H_{*+1}(E_Y^- S_U(\hat C)) \arrow[r, "E_Y S_U(p)"] & H_*(E_Y^- S_U(\bar C)) \arrow[r, "E_Y S_U(i)"] & H_*(E_Y^- S_U(\check C)) \cdots
\end{tikzcd}$$

To see this, note that in the third column of (5.16), the map

$$E_Y S_u(p)\colon\ H_*(E_Y^\infty S_U(\hat C)) \to H_*(E_Y^\infty S_U(\bar C))$$

is an isomorphism by the preceding Lemma and Proposition 4.8. Therefore $H_*(E_Y^\infty S_U(\check C))$ is trivial. This in turn implies that the map

$$\delta_{V*}\colon\ H_{*+1}(E_Y^+ S_U(\check C)) \to H_*(E_Y^- S_U(\check C))$$

in the third row of (5.16) is an isomorpism. For the same reasons, the map

$$i_{V_*}\colon\ H_*(E_Y^- S_U(\bar C)) \to H_*(E_Y^\infty S_U(\bar C))$$

on the first and fifth rows of (5.16) is an isomorphism as well, and therefore $H_*(E_Y^+ S_U(\bar C))$ is trivial too. This in turn implies that the map

$$E_Y S_U(j)\colon\ H_*(E_Y^+ S_U(\check C)) \to H_*(E_Y^+ S_U(\hat C))$$

on the first and fourth columns is an isomorphism. We now take:

$$\hat h = \mathrm{Id}, \quad \bar h = i_{V*}^{-1} \circ E_Y S_U(p), \quad \check h = \delta_{V_*} \circ (E_Y S_U(j))^{-1},$$

and the Proposition follows. □

Remark 5.10 By Theorem 5.1 and Lemmas 4.4, 4.7, the chain complex $\hat C$ in the statement of the preceding proposition may be replaced by $CM(M, \mathfrak{s}, c_-)$ (yet $\mathring{HM}$ still stands for $\mathring{HM}(M, \mathfrak{s}, c_b)$).

6 Monopole Floer homology under connected sum

We follow the (by now) traditional approach to connected sum formula for Floer homologies that appeared in the instanton Floer homology setting in [F, D]. Some setting-up is required.

6.1 Preparations

Let M_1, M_2 be closed, oriented, connected 3-manifolds and $\mathfrak{s}_1$, $\mathfrak{s}_2$ be spin-c structures on M_1, M_2 respectively. Denote by $M_\sqcup = M_1 \sqcup M_2$ the disjoint union of M_1, M_2. Let $\mathfrak{s}_\sqcup$ denote the Spin^c structure on $M_\sqcup$ given by $\mathfrak{s}_1$ and $\mathfrak{s}_2$.

Part 1: Spin-c structures and gradings. Recall from [KM] the interpretation of spin-c structures and grading via oriented 2-plane fields on the 3-manifold M. Denote by $\mathbb{J}(M)$ the set of homotopy classes of oriented 2-plane fields on the 3-manifold M. According to Proposition 23.1.8 of [KM], this may be identified with the set of gradings of the manifold M, as defined in [KM] p.424. There is a $\mathbb{Z}$ action on $\mathbb{J}(M)$, defined by modifying a representing plane field in a ball in M ([KM], Definition 3.1.2). Its quotient is the set of spin-c structures over M, $\mathrm{Spin}\,(M) = \mathbb{J}(M)/\mathbb{Z}$. The orbit over $\mathfrak{s} \in \mathrm{Spin}\,(M)$ is the set of gradings for the spin-c structure $\mathfrak{s}$, which we denote by $\mathbb{J}(M, \mathfrak{s})$. Let $c_\mathfrak{s}$ be the divisibility of $c_1(\mathfrak{s})$. The stabilizer of the orbit $\mathbb{J}(M, \mathfrak{s})$ is $c_\mathfrak{s}\mathbb{Z}$; therefore $\mathbb{J}(M, \mathfrak{s})$ is a torsor under $\mathbb{Z}/c_\mathfrak{s}\mathbb{Z}$.

Let $B(p_1)$, $B(p_2)$ be respectively open balls centered at $p_1 \in M_1$, $p_2 \in M_2$, and $\varphi\colon B(p_1)\backslash\{p_1\} \to B(p_2)\backslash\{p_2\}$ be an orientation reversing map such that

$$M_\# := M_1 \# M_2 := (M_1\backslash\{p_1\} \cup M_2\backslash\{p_2\})/\sim_\varphi .$$

As described in [KM], the $\mathbb{Z}$ action on $\mathbb{J}(Y_i)$ is induced from the $C^0(Y_i, SO(3))$-action on the space of plane fields on Y_i. Each element in the group $\mathbb{Z}$ is represented by an even-degree element in $C^0(Y_i, SO(3))$ sending $Y_i\backslash B(p_i)$ to $1 \in SO(3)$. Since this map has even degree, we may choose it to send $p_i \in B(p_i)$ to 1 as well. The orientation reversing map φ then defines an isomorphism:

$$\iota_\mathbb{J}\colon \big(\mathbb{J}(M_1) \times \mathbb{J}(M_2)\big)/\bar{\Delta} \to \mathbb{J}(M_\#),$$

where $\bar{\Delta} \subset \mathbb{Z} \times \mathbb{Z}$ denotes the anti-diagonal. This isomorphism is equivariant with respect to the residual $\mathbb{Z}$ action on $\iota_\mathbb{J}\colon \big(\mathbb{J}(M_1) \times \mathbb{J}(M_2)\big)/\bar{\Delta}$ and the $\mathbb{Z}$ action of $\mathbb{J}(M_\#)$. Thus, by taking the quotient on both sides above, one has an induced isomorphism

$$\iota_S\colon \ \mathrm{Spin}(M_1) \times \mathrm{Spin}(M_2) \to \mathrm{Spin}(M_\#).$$

Let $\mathfrak{s}_\# = \iota_S(\mathfrak{s}_1, \mathfrak{s}_2)$.

Note that restrictions of $\iota_\mathbb{J}$ to orbits of the $\mathbb{Z}$ actions give rise to isomorphisms from $(\mathbb{J}(M_1, \mathfrak{s}_1) \times \mathbb{J}(M_2, \mathfrak{s}_2))/\bar{\Delta}$ to $\mathbb{J}(Y, \mathfrak{s}_\#)$, both of which are affine spaces under $\mathbb{Z}/c_\#\mathbb{Z}$, where $c_\#$ is the g.c.d of $c_{\mathfrak{s}_1}$ and $c_{\mathfrak{s}_2}$.

Remark 6.1 Note that the canonical $\mathbb{Z}/2\mathbb{Z}$-grading ([KM] § 22.4) of $\iota_\mathbb{J}(\xi_1, \xi_2)$ differs from the sum of the canonical $\mathbb{Z}/2\mathbb{Z}$ grading of ξ_1 and ξ_2 by 1.

Part 2: The cobordism X. Let $\mathcal{V} := (X, s)$ denote a cobordism described in (2.7) and (2.8), with $Y_- = M_\#$ and $Y_+ = M_\sqcup$. Assume that s has a unique critical point of index 3 with critical value 0.

There is a unique Spin^c structure $\mathfrak{s}_X$ on X with $c_1(\mathfrak{s}_X)|_{s^{-1}(-c)} = c_1(\mathfrak{s}_\#)$ and $c_1(\mathfrak{s}_X)|_{s^{-1}(c)} = c_1(\mathfrak{s}_\sqcup)$ for $c \gg 0$. Meanwhile, given $[\varpi_i] \in H^2(Y_i)$, there is a unique $[\varpi_\#] \in H^2(M_\#)$ and a $[\varpi_X] \in H^2(X)$ that restricts to ϖ_1, ϖ_2, $\varpi_\#$ respectively on the M_1, M_2, $M_\#$ ends of X.

Let $\overline{\mathcal{V}} := (X, -s)$ denote the "time-reversal" of $\mathcal{V}$. Given local systems Γ_i on $\mathcal{B}^\sigma(Y_i)$, $i = 1, 2$, let $\Gamma_\sqcup = \Gamma_1 \otimes \Gamma_2$ denote the local system on $\mathcal{B}^\sigma(M_1) \times \mathcal{B}^\sigma(M_2) \simeq \mathcal{B}^\sigma(M_\sqcup)$. As the cobordism $\overline{\mathcal{V}} = (X, -s)$ corresponds to attaching a 1-handle to $M_\sqcup$ and $\mathcal{V} = (X, s)$ to attaching a 3-handle to $M_\#$, a local system $\Gamma_\sqcup$ on $B^\sigma(M_\sqcup)$ determines a local system $\Gamma_\#$ on $\mathcal{B}^\sigma(M_\#)$ and a $\overline{\mathcal{V}}$-morphism from $\Gamma_\sqcup$ to $\Gamma_\#$ in the sense of [KM] Definition 23.3.1.Conversely, a $\Gamma_\#$ determines a local system $\Gamma_\sqcup$ on $B^\sigma(M_\sqcup)$ and a $\mathcal{V}$-morphism from the former to the latter.

6.2 A connected sum formula for non-balanced perturbations

Let $\mathcal{V} = (X, s)$ and $\overline{\mathcal{V}} = (X, -s)$ be as described in Part 2 of the previous subsection, and suppose the two-form ϖ_X satisfies (2.10) X satisfies (2.17). Recall the definition of $(\hat{C}(M_\sqcup), \hat{\partial}_{M_\sqcup})$ and $\hat{U}_{M_\sqcup}$ in this situation from the end of Section 2.6. Recall also the maps $m^\sharp_{o\natural}[u]$ associated to the cobordism $\mathcal{V}$, and the maps $m^{o\sharp}_\natural[u]$ associated to $\overline{\mathcal{V}}$. As explained in the previous subsection, the monopole Floer chain complex $\hat{C}(M_\sqcup)$ is graded by $(\mathbb{J}(M_1, \mathfrak{s}_1) \times \mathbb{J}(M_2, \mathfrak{s}_2))/\overline{\Delta}$.

Proposition 6.2 *Under the above assumptions:*

(a) *Suppose that* $\mathrm{r}[w_\#]$ *is negative monotone, non-balanced with respect to* $\mathfrak{s}_\#$*. Then the following two (relatively)* $\mathbb{Z}/c_\#$*-graded modules over*

$$\mathbf{A}_\dagger(M_\#) \simeq \textstyle\bigwedge^*(H_1(M_1)/Tors) \otimes \bigwedge^*(H_1(M_2)/Tors) \otimes \mathbb{K}[u]$$

are chain-homotopic:

$$C_*\big(M_\#, \mathfrak{s}_\#, \mathrm{r}[w_\#], \Gamma_\#\big) \quad \textit{and} \quad S_{U_\sqcup}\big(\hat{C}_*(M_\sqcup, \mathfrak{s}_\sqcup, \mathrm{r}[w]_\sqcup; \Gamma_\sqcup)\big). \tag{6.1}$$

(b) *Suppose that* $[w_1]$ *is nonbalanced with respect to* $\mathfrak{s}_1$*, and* Γ_i *is strongly* $(\mathfrak{s}_i, [w_i])$*-complete for* $i = 1, 2$*. Then the following two (relatively)* $\mathbb{Z}/c_\#$*-graded modules over* $\mathbf{A}_\dagger(M_\#)$ *are chain-homotopic:*

$$C_\bullet\big(M_\#, \mathfrak{s}_\#, [\varpi_\#], \Gamma_\#\big) \quad \textit{and} \quad S_{U_\sqcup}\big(\hat{C}_\bullet(M_\sqcup, \mathfrak{s}_\sqcup, [\varpi_\sqcup]; \Gamma_\sqcup)\big).$$

Proof. **Part (a):** Use the short-hand $C_\sqcup = C(M_\sqcup)$, $C_\# = C(M_\#)$ etc in this proof. Write $S_{U_\sqcup}(\hat{C}_\sqcup)$ as the direct sum:

$$S_{U_\sqcup}(\hat{C}_\sqcup) = \hat{C}_\sqcup \oplus y\,\hat{C}_\sqcup. \tag{6.2}$$

With respect to this decomposition, it differential takes the following block form:

$$D_\sqcup = \begin{bmatrix} \hat{\partial}_\sqcup & 0 \\ \hat{U}_\sqcup & -\hat{\partial}_\sqcup \end{bmatrix}. \tag{6.3}$$

Here, we take U_1 and $\hat{U}_2$ used in the definition of $\hat{U}_\sqcup$ to be given by the holonomies along p_1 and p_2 respectively, as described in the beginning of Part 2 in Section 2.4. The proof has three steps.

Step 1. Recall the cobordisms $\mathcal{V}$, $\overline{\mathcal{V}}$ from last subsection. Let λ denote the ascending manifold of the unique critical point of s; it is a path in X asymptotic to $(p_1, p_2) \in M_\sqcup = Y_+$. We orient it so that it begins from $p_1 \in M_1$ and ends at $p_2 \in M_2$. Meanwhile, the descending manifold of this critical point will be denoted B; it is an embedded 3-ball in X that intersects each $s^{-1}(c) \simeq M_\#$ in a 2-sphere, $\forall c \ll 0$. Let $\overline{\lambda}$, $\overline{B}$ respectively denote the descending and ascending manifold from the unique critical point of $-s$. These are the same submanifolds in X as λ, B, but equipped with the opposite orientation.

Let w_X be a 2-form on X satisfying (2.11). We now describe how the triple $\mathcal{V}, \mathrm{r}w_X, \lambda$ defines a chain map V_* from $\hat{C}(M_\#)$ to $S_{U_\sqcup}(\hat{C}_\sqcup)$, and conversely, how $\overline{\mathcal{V}}, \mathrm{r}w_X, \overline{\lambda}$ defines a chain map $V_*^\dagger$ from the latter to the former.

Corresponding to the decomposition (6.3), these maps have the block form

$$V_* = \begin{bmatrix} V_0 \\ V_1 \end{bmatrix}, \quad V_*^\dagger = \begin{bmatrix} V_1^\dagger & V_0^\dagger \end{bmatrix}. \tag{6.4}$$

Here, V_i, $V_i^\dagger$, $i = 0, 1$, are respectively defined from Seiberg-Witten moduli spaces $\mathcal{M}_i(X, \mathfrak{s}_X, \mathrm{r}w_X)$ described in Section 3.2 as follows:

$$\begin{gathered} V_0 = \hat{m}[1](\mathcal{V}), \quad V_1 = \hat{m}[u_\lambda](\mathcal{V}), \\ V_1^\dagger = \hat{m}[u_{\overline{\lambda}}](\overline{\mathcal{V}}), \quad V_0^\dagger = \hat{m}[1](\overline{\mathcal{V}}), \end{gathered} \tag{6.5}$$

where $1 \in C^0(\mathcal{U})$ is the constant function 1, $u_\lambda = \delta\,\mathrm{hol}_\lambda \in C^1(\mathcal{U})$, with hol_λ denoting the holonomy along λ. The explicit formulae for $\hat{m}[u](\mathcal{V})$, $\hat{m}[u](\overline{\mathcal{V}})$ are given in the end of Section 2.6, with local coefficients incorporated, as:

$$\begin{aligned} &\hat{m}[u](\mathcal{V})(\mathfrak{c}_\#) = \\ &\sum_{(\mathfrak{c}_1,\mathfrak{c}_2)\in\mathfrak{C}(M_1)\times\mathfrak{C}(M_2)} \; \sum_{z\in\pi_0(\mathcal{B}^\sigma(\mathcal{V};\mathfrak{c}_\#,(\mathfrak{c}_1,\mathfrak{c}_2)))} \Gamma_{X,z}\big(\langle u, \mathcal{M}_{k,z}(\mathfrak{c}_\#, (\mathfrak{c}_1,\mathfrak{c}_2))\rangle \Gamma_\sqcup(\mathfrak{c}_1,\mathfrak{c}_2)\big) \end{aligned} \tag{6.6}$$

and its analog for $\hat{m}[u](\overline{\mathcal{V}})$. In the above expression, $\mathcal{M}_{k,z}(\mathfrak{c}_\#, (\mathfrak{c}_1, \mathfrak{c}_2)) \subset \mathcal{M}_k(\mathfrak{c}_\#, (\mathfrak{c}_1, \mathfrak{c}_2))$ denotes the subset consisting of elements with relative homotopy class z.

To see that the right hand side of (6.6) is well-defined under the assumptions of Part (a), observe the following: By the well-known compact property of spaces of 3-dimensional Seiberg-Witten solutions, $CM(M_1) = C^o(M_1)$, $C^o(M_2)$, $CM(M_\#) = C^o(M_\#)$ are all finitely generated over $\mathbb{K}$, while $C^u(M_2)$ is finitely generated over $\mathbb{K}[u]$, with u having degree -2. Write the generating sets of these free $\mathbb{K}$-modules respectively as $\mathfrak{C}(M_1) = \{\mathfrak{a}_i\}_i$, $\mathfrak{C}^o(M_2) = \{\mathfrak{b}^o_j\}_j$, $\mathfrak{C}(M_\#) = \{\mathfrak{c}_k\}_k$, and $\mathfrak{C}^u(M_2) = \{\mathfrak{b}^u_q u^n\}_{q,n}$, where there are finitely many indices i, j, k, q, and n runs through all non-negative integers. Let $\pi^\sigma \colon \mathcal{B}^\sigma \to \mathcal{B}$ denote the projection of the blown-up space. The index $\imath_{\mathfrak{d}}$ and the topological energy ([KM] Definition 4.5.4) of an element $\mathfrak{d} \in \mathcal{M}(\mathcal{V})$ depends only its relative homotopy class under π^σ, and the former is controlled via $c_1(\mathfrak{s}_X)$, the latter through $[\mathrm{r}w_X] - \pi[c_1(\mathfrak{s}_X)]$. The monotonicity condition and the compactness property of $\mathcal{M}(\mathcal{V})$ under bounds on the topological energy then ensures that only finitely $\mathfrak{a}_i$, $\mathfrak{b}^o_j$, $\mathfrak{b}^u_q$, z appear in the sum on the right hand side of (6.6). Meanwhile, since $\mathrm{gr}(\mathfrak{b}^u_q u^n) - \mathrm{gr}(\mathfrak{b}^u_q u^m) = -2(n-m)$, the index bound $\imath_{\mathfrak{d}} = k$ on the right hand side of (6.6) implies that for each q, only finitely many $\mathfrak{b}^u_q u^n$ appears on the right hand side of (6.6). (The aforementioned compactness result follows from a straightforward generalization of Theorem 24.5.2 in [KM] to include nonexact perturbations).

Meanwhile, the $\hat{m}(\overline{\mathcal{V}})$ analog of (6.6) involves sum over $\mathfrak{C}(M_\#)$ instead, which consists of finitely many elements. The finiteness of the relevant sum then follows from the monotonicity assumption alone.

Step 2. In this step, we show that V_*, $V_*^\dagger$ are well defined chain maps. This amounts to verifying the following identities:

$$
\begin{aligned}
\hat{\partial}_\sqcup V_0 + V_0 \partial_\# &= 0 \\
\hat{\partial}_\sqcup V_1 - V_1 \partial_\# - \hat{U}_\sqcup V_0 &= 0 \\
\partial_\# V_0^\dagger - V_0^\dagger \hat{\partial}_\sqcup &= 0, \\
\partial_\# V_1^\dagger + V_1^\dagger \hat{\partial}_\sqcup + V_0^\dagger \hat{U}_\sqcup &= 0.
\end{aligned}
\tag{6.7}
$$

To see about the signs in (6.7), note that $H^*(V, M_\sqcup)$, $H^*(V^\dagger, M_\#)$ are respectively 1 and 0 dimensional. Thus the homology orientation convention of Definition 3.4.1 of [KM] dictates that $\langle [\mathfrak{c}], m^{os}_o([\mathfrak{a}], [\mathfrak{b}]) \rangle$ should abstractly be regarded as a homomorphism from $\lambda_{[\mathfrak{a}]} \wedge \lambda_{[\mathfrak{b}]}$ to $\mathbb{K} \wedge \lambda_{[\mathfrak{c}]}$, and $\langle ([\mathfrak{a}], [\mathfrak{b}]), m^o_{ou}[\mathfrak{c}] \rangle$ regarded as one from $\lambda_{[\mathfrak{c}]}$ to $\lambda_{[\mathfrak{a}]} \wedge \lambda_{[\mathfrak{b}]}$. They are identifiable with numbers after orientations of $\lambda_{[\mathfrak{a}]}$, $\lambda_{[\mathfrak{b}]}$, $\lambda_{[\mathfrak{c}]}$, and $H^*(V, M_\sqcup)$

are chosen. Here, $\lambda_{[*]}$ etc denote the orientation line bundle from the critical point $[*]$. Thus, V_0 and $V_1^\dagger$ are odd maps, while V_1 and $V_0^\dagger$ are even.

To verify (6.7), write the identities in full in terms of $m^o_{o\natural}$, $m_o^{o\sharp}$, $\partial^o_o(M_1)$, $\partial^o_o(M_\#)$, $\hat\partial(M_2)$, $\overline{\partial}{}^s_u(M_2)$ as given by (2.14), (6.6), (6.4). These can be reduced to the identities in Lemma 25.3.6 in [KM] (with many vanishing terms), with these substitutions:

- Drop the "o"'s from the double superscript or subscripts $o*$ of m,
- Replace the entries of $\hat\partial(M_\sqcup) = (1\otimes\partial^\sharp_\natural(M_2) + \partial^o_o(M_1)\otimes 1))$ by $\partial^\sharp_\natural$.

Theorem 24.7.2 in [KM] conveniently supplies us with the general gluing theorem required for verifying these formulae. (We have at worst rank 1 boundary-obstruction).

Step 3. In this step, we show that the two chain complexes in (6.1) are chain-homotopy equivalent via V_* and $V_*^\dagger$. In other words, their compositions satisfy the following identities:

$$\begin{aligned}
V_*^\dagger\circ V_* &= V_1^\dagger\circ V_0 + V_0^\dagger\circ V_1\\
&= \mathrm{Id}_\# + [\partial_\#, H_\#],\\
V_*\circ V_*^\dagger &= \begin{bmatrix} V_0\circ V_1^\dagger & V_0\circ V_0^\dagger\\ V_1\circ V_1^\dagger & V_1\circ V_0^\dagger\end{bmatrix}\\
&= \begin{bmatrix}\mathrm{Id}_\sqcup & 0\\ 0 & \mathrm{Id}_\sqcup\end{bmatrix} + \left[\begin{bmatrix}\hat\partial_\sqcup & 0\\ \hat U_\sqcup & -\hat\partial_\sqcup\end{bmatrix}, \begin{bmatrix} A & B\\ C & D\end{bmatrix}\right].
\end{aligned}\tag{6.8}$$

To verify these identities, consider the composed cobordism $W_\#$ of $\mathcal{V}$ with $\overline{\mathcal{V}}$. This cobordism goes from $M_\#$ to $M_\#$, and contains the circle $\lambda_\# = \lambda\cup\overline{\lambda}$ in its interior. A surgery along $\lambda_\#$ replacing a tubular neighborhood $S^1\times D^3$ of $\lambda_\#$ with $D^2\times S^2$ yields $\mathbb{R}\times M_\#$. On the other hand, compose in the opposite order to get the cobordism $W_\sqcup$ from $M_\sqcup$ to $M_\sqcup$. There is an embedded 3-sphere $S_\sqcup\subset W_\sqcup$ obtained by joining B and $\overline{B}$. Doing a surgery along $S_\sqcup$, namely, replace a tubular neighborhood of it, $I\times S_\sqcup$, by a disjoint union of two 3-balls $B_1\sqcup B_2$, turns $W_\sqcup$ into the product cobordism $\mathbb{R}\times M_\sqcup$. One may find arcs $\lambda_1\subset B_1$, $\lambda_2\subset B_2$ so that under this surgery they joint with $(\lambda\cup\overline{\lambda}) - I\times S_\sqcup$ to yield $\mathbb{R}\times\{p_1,p_2\}\subset\mathbb{R}\times M_\sqcup$.

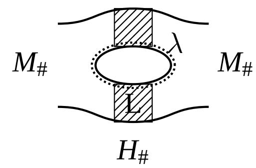
It follows from analogs of Proposition 26.1.2 in [KM] that:

$$(6.9)\quad \begin{cases} 1)\ \text{The map } V_1^\dagger V_0 + V_0^\dagger V_1 = m[u_{\lambda_\#}](W_\#) + [\partial_\#, H_\#]. \\ 2)\ \text{The map } V_0 V_1^\dagger = \hat{m}[u_\lambda^-](W_\sqcup) + [\hat{\partial}_\sqcup, A] - B\hat{U}_\sqcup. \\ 3)\ \text{The map } V_0 V_0^\dagger = \hat{m}[1](W_\sqcup) + [\hat{\partial}_\sqcup, B]. \\ 4)\ \text{The map } V_1 V_1^\dagger = \hat{m}[u_\lambda^- u_\lambda](W_\sqcup) - [\hat{\partial}, C] + \hat{U}_\sqcup A - D\hat{U}_\sqcup. \\ 5)\ \text{The map } V_1 V_0^\dagger = \hat{m}[u_\lambda](W_\sqcup) - [\hat{\partial}_\sqcup, D] + \hat{U}_\sqcup B. \end{cases}$$

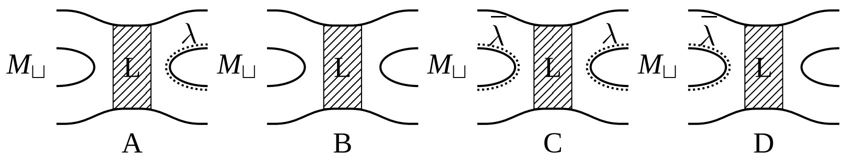
To say more, recall the proof of Proposition 26.1.2 in [KM]. This is based on a gluing argument for composing cobordisms.

Though Proposition 26.1.2 concerns only cobordisms between connected 3-manifolds while Statement (1) above involves gluing along $M_\sqcup$, the condition that only the M_2 component can be associated with balanced perturbations imply that the straightforward sort of gluing argument may be used to glue along M_1, leaving the more delicate analysis described in [KM] required for M_2 alone. In the case discussed in [KM], the chain homotopy is constructed from a parametrized moduli space for manifolds obtained by inserting a "long neck" between the composed cobordism. See (26.2), (26.3). The relevant chain homotopy is defined in (26.12) and the identity it satisfies is proved in Lemma 26.2.2 of [KM]. The aforementioned parametrized moduli space is compactified so as to form a cobordism between the moduli space associated to the manifold with "neck length 0" and that with "neck length ∞". In our context, the chain maps associated to the cobordism with "neck length 0" corresponds to $m[u_{\lambda_\#}](W_\#)$, $\hat{m}[u_\lambda^-](W_\sqcup)$, $\hat{m}[1](W_\sqcup)$, $\hat{m}[u_\lambda^- u_\lambda](W_\sqcup)$, $\hat{m}[u_\lambda](W_\sqcup)$. The moduli space associated with cobordism with "neck length ∞" is a union of spaces of "broken trajectories". These give rise to the remaining terms in (6.9). To be more specific, the maps $H_\#$, A, B, C, D are respectively analogs of the map $\hat{K}$ in (26.12) of [KM], given by parametrized moduli spaces for manifolds illustrated in Figures 6.10, 6.11 below. The identities they satisfy, namely (6.9), are illustrated in Figures 6.12-6.16. Here, the dotted arcs represents the arcs over which holonomy is taken over to define the Čech 1-cocyles used in the maps. The slashed lines correspond to composition of maps in (6.9); geometrically, they corresponds to where the "trajectory" is broken and gluing takes place. The cobordisms with shaded regions are associated with maps defined via parametrized moduli spaces, where the parameter is the "neck length" L of the shaded region.

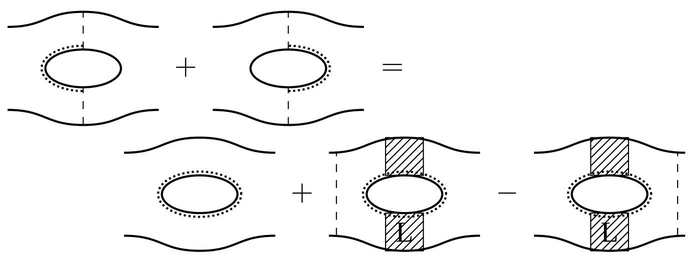
(6.10)



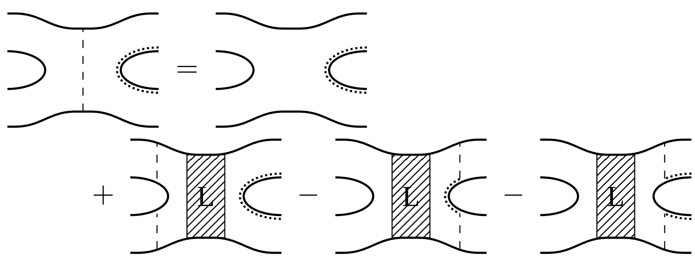
(6.11)



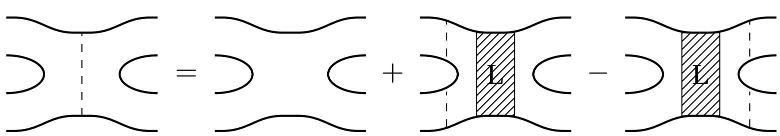
(6.12)



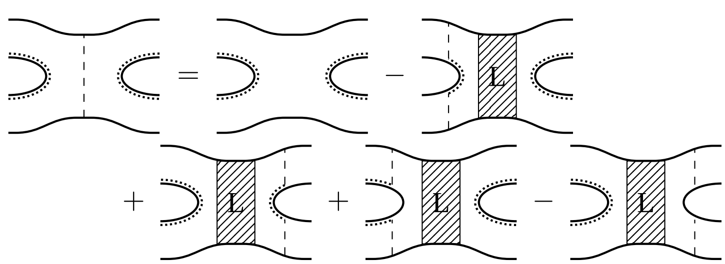
(6.13)



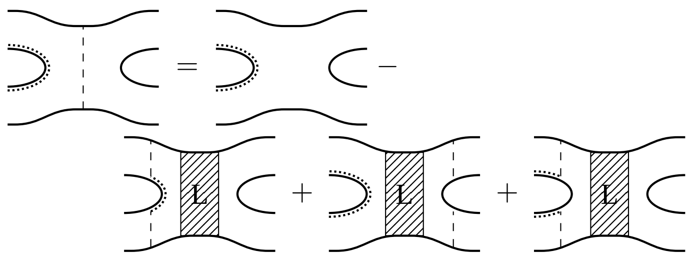
(6.14)



(6.15)



(6.16)



Next, we study the cobordism maps $m[u_{\lambda_\#}](W_\#)$, $\hat{m}[u_{\bar{\lambda}}](W_\sqcup)$, $\hat{m}[1](W_\sqcup)$, $\hat{m}[u_{\bar{\lambda}} u_\lambda](W_\sqcup)$, $\hat{m}[u_\lambda](W_\sqcup)$ in (6.9) by surgering along $\lambda_\# \subset W_\#$ and $S_\sqcup \subset W_\sqcup$ as described above. Then, standard gluing theorems for 4-manifolds cut along compact, oriented 3-manifolds (either $S^1 \times S^2$ or S^3 in our setting) imply that they are respectively $\mathrm{Id}_\#$, $\mathrm{Id}_\sqcup$, 0, 0, $\mathrm{Id}_\sqcup$. As this type of gluing theorem is well-documented in the Seiberg-Witten literature, and their Yang-Mills analogs are given in Theorem7.16 and Corollary 7.21 of [D], only brief sketches of their proofs are given below.

(1) About $m[u_{\lambda_\#}](W_\#)$: Let $W'_\# = W_\# - U(\lambda_\#)$, $U(\lambda_\#) \simeq S^1 \times D^3$ being a tubular neighborhood of $\lambda_\#$. As previously explained, $\mathbb{R} \times M_\# \simeq W'_\# \cup_{S^1 \times S^2} (D^2 \times S^2)$. Insert a long neck $[-L, L] \times S^1 \times S^2$ with the standard metric along the embedded $S^1 \times S^2$ in $W_\#$ and $\mathbb{R} \times M_\#$, and choose the metric on $U(\lambda_\#) \subset W_\#$ and $D^2 \times S^2 \subset \mathbb{R} \times M_\#$ to have nonnegative scalar curvature. According to (2.15), the coefficients of $m[u_{\lambda_\#}](W_\#)$ are given by the numbers

$$\langle u_{\lambda_\#}, \mathcal{M}_1(W_\#; \mathfrak{c}_-, \mathfrak{c}_+)\rangle \quad \text{for } \mathfrak{c}_-, \mathfrak{c}_+ \in \mathcal{Z}_{w_\#, \mathrm{r}}(M_\#).$$

A typical gluing argument yields that when L is large enough, $\mathcal{M}_1(W_\#; \mathfrak{c}_-, \mathfrak{c}_+)$ has the structure of a fibered product $\mathcal{M}_1(W'_\#; \mathfrak{c}_-, \mathfrak{c}_+) \times_{\mathcal{Z}(S^1 \times S^2)} \mathcal{M}(S^1 \times D^3)$. Here, $\mathcal{Z}(S^1 \times S^2)$ is the moduli of solutions to (2.1) with $M = S^1 \times S^2$, $c_1(\mathfrak{s}) = 0$, $[\varpi] = 0$. This consists of a circle of flat connections, parametrized by the value of their holonomy along $S^1 \times \{p\}$. Meanwhile, the fibering map $\mathcal{M}(S^1 \times D^3) \to \mathcal{Z}(S^1 \times S^2)$ is a diffeomorphism. Thus, $\langle u_{\lambda_\#}, \mathcal{M}_1(W_\#; \mathfrak{c}_-, \mathfrak{c}_+)\rangle$ is the Euler characteristic of a fiber of $\mathcal{M}_1(W'_\#; \mathfrak{c}_-, \mathfrak{c}_+)$. Meanwhile, $\mathcal{M}(D^2 \times S^2)$ (with infinitely long neck) consists of a single flat connection sitting over the element in $\mathcal{Z}(S^1 \times S^2)$ corresponding to zero holonomy. Thus, for sufficiently long neck, $\mathcal{M}_1(\mathbb{R} \times M_\#; \mathfrak{c}_-, \mathfrak{c}_+) \simeq \mathcal{M}_1(W'_\#; \mathfrak{c}_-, \mathfrak{c}_+) \times_{\mathcal{Z}(S^1 \times S^2)} \mathcal{M}(D^2 \times S^2)$ is homeomorphic to a fiber of $\mathcal{M}_1(W'_\#; \mathfrak{c}_-, \mathfrak{c}_+)$ as well. Thus, $\langle 1, \mathcal{M}_1(\mathbb{R} \times M_\#; \mathfrak{c}_-, \mathfrak{c}_+)\rangle$ is simply the Euler characteristic of the latter.

(2) About $\hat{m}[u_{\bar{\lambda}}](W_\sqcup)$: Write $W_\sqcup$ is a connected sum of $W_1 \simeq \mathbb{R} \times M_1$ and $W_2 \simeq \mathbb{R} \times M_2$ at interior points. Thus, (after inserting a long neck $[-L, L] \times S^3$ at the point

of connected sum), $\mathcal{M}(W_\sqcup)$ has the structure of $\big(\tilde{\mathcal{M}}(W_1)\times\tilde{\mathcal{M}}(W_2)\big)/S^1$, where $\tilde{\mathcal{M}}(W_i)$ is the framed moduli space, upon which there is a S^1-action such that the orbit space is $\mathcal{M}(W_i)$. In the above product space, S^1 acts diagonally. The arc $\bar\lambda$ is divided at the point of connected sum into $\bar\lambda_1\cup\bar\lambda_2$. Thus, $\langle u_{\bar\lambda}^-, \mathcal{M}_1(W_\sqcup;\mathfrak{c}_\sqcup,\mathfrak{c}'_\sqcup)\rangle$ is computed by

$$\begin{aligned}&\langle \pi^* u^-_{\bar\lambda_1}, \tilde{\mathcal{M}}_1(W_1';\mathfrak{c}_1,\mathfrak{c}_1')\rangle\cdot\langle 1,\mathcal{M}_0(W_2';\mathfrak{c}_2,\mathfrak{c}_2')\rangle\\ &=\langle 1,\mathcal{M}_0(W_1';\mathfrak{c}_1,\mathfrak{c}_1')\rangle\cdot\langle \pi^* u^-_{\bar\lambda_2}, \tilde{\mathcal{M}}_1(W_2';\mathfrak{c}_2,\mathfrak{c}_2'\rangle\\ &=\langle 1,\mathcal{M}_0(W_1;\mathfrak{c}_1,\mathfrak{c}_1')\rangle\cdot\langle 1,\mathcal{M}_0(W_2;\mathfrak{c}_2,\mathfrak{c}_2')\rangle,\end{aligned}$$

where $\pi\colon\ \tilde{\mathcal{M}}(W_i)\to\mathcal{M}(W_i)$ is the quotient map, W_i' is W_i with a point removed (and with a complete metric equipping it with an infinitely long neck), $\mathfrak{c}_\sqcup=\mathfrak{c}_1\otimes\mathfrak{c}_2$ and $\mathfrak{c}'_\sqcup=\mathfrak{c}'_1\otimes\mathfrak{c}'_2$. The last line of the above expression is $\langle 1,\mathcal{M}_0(\mathbb{R}\times M_\sqcup;\mathfrak{c}_\sqcup,\mathfrak{c}'_\sqcup)\rangle$. and glue back two 4-balls. This gluing argument would show that the induced map on Floer homologies is the same as that defined from $\mathbb{R}\times Y_\sqcup$.

(3) About $\hat{m}[1](W_\sqcup)$: The space of glued solutions have expected dimension $1\neq 0$.

(4) About $\hat{m}[u_{\bar\lambda}^-u_\lambda](W_\sqcup)$: Similar to (3); The space of glued solutions have expected dimension $1\neq 2$.

(5) About $\hat{m}[u_\lambda](W_\sqcup)$: Similar to (2).

Lastly, the statement concerning module structures in the proposition follows from the fact that the isomorphism (6.1) is defined by cobordisms $\mathcal{V}$, $\mathcal{V}^\dagger$, and maps between monopole Floer homologies defined by any given cobordism X has the structure of a module over $\bigwedge^*(H_1(X)/\mathrm{Tors})\otimes\mathbb{K}[U]$. (See Theorem 3.4.4 in [KM]). In our case,

$$H_1(X)=H_1(M_\sqcup)=H_1(M_\#).$$

Part (b): The proof for Assertion (b) of this proposition differs from part (a) only in the mechanism to ensure that the right hand side of (6.6) and its analog are well-defined. Instead of monotonicity, this is now justified by the completeness condition on the local coefficients, and by working with the grading-completed version of monopole Floer complexes $C_\bullet$. The relevant compactness theorem here is Theorem 24.5.2 of [KM]. □

Remark 6.3 (a) By Theorem 5.1 again, the monopole Floer chain complex in the statement of the preceding can be associated to any suitable $(\mathfrak{T},\mathfrak{S})$.

(b) Note that when $c_1(\mathfrak{s})$ is torsion, the following types of perturbations are all equivalent: positive monotone, negative monotone, balanced, exact. Thus, the assumption in

part (a) implies that $c_1(\mathfrak{s}_\#)$ is nontorsion. On the other hand, the assumption that $[w_\#]$ is monotone with respect to $c_\#$ in part (a) implies that both $[w_1]$, $[w_2]$ are respectively monotone with respect to $c_1(\mathfrak{s}_1)$, $c_1(\mathfrak{s}_2)$ with the same monotonicity constant. Combined with the assumption that $[w_\#]$ is nonbalanced with respect to $c_1(\mathfrak{s})$, this implies that $[w_i]$ is nonbalanced with respect to $c_1(\mathfrak{s}_i)$ for at least one of $i=1$ or 2. We shall always choose M_1 to be the one endowed with a nonbalanced perturbation.

(c) Our proof follows the "standard" cobordism argument that appeared in [F] and [D] §7.4 in the Yang-Mills setting. Bloom-Mrowka-Ozsvath [BMO] proved a connected sum formula for exact perturbations using a different approach that involves surgery exact sequences. The use of the latter necessitates the use of the completed version of monopole Floer homologies $HM_\bullet$.

6.3 Filtered monopole Floer homology and handle addition

Continue to work with the same settings and notation from earlier parts of the section, but now specialize to the 3-manifolds and cobordisms described in Sections 3.6, 3.7. More specifically, the following two cases are considered: Fix an $\mathrm{r} \gg \pi$.

(6.17)

1) Let $M_1 = Y_i$, for $i = 0, \ldots, \mathrm{G}-1$. Equip Y_i with the nontorsion Spin^c structure and a metric from the set *Met* in Proposition 3.8. Let w_1 be the corresponding harmonic 2-form w in Proposition 3.8. Let $M_2 = S^1 \times S^2$, $\mathfrak{s}_2$ be the trivial Spin^c structure, and $w_2 \equiv 0$. Then $M_\# \simeq Y_{i+1}$, and $[w_\#] = c_1(\mathfrak{s}_\#)$ is nontorsion. Choose the metric on Y_{i+1} to be from the set *Met* from Proposition 3.8.

2) Let $M_1 = S^1 \times S^2$, with the nontorsion Spin^c-structure $\mathfrak{s}_1$, closed 2-form w_1, and metric as described in Part 1 of Section 3.6. Let $(M_2, \mathfrak{s}_2) = (M, \mathfrak{s})$ be a connected Spin^c 3-manifold, with $\varpi_2 = \mathrm{r} w_2$ for a closed 2-form w_2 in the cohomlogy class $c_1(\mathfrak{s}_2)$. Choose a metric on M with respect to which w_2 is harmonic, and in the case when $c_1(\mathfrak{s})$ is non-torsion, having nondegenerate zeros. (When $c_1(\mathfrak{s})$ is torsion, w_2 is necessarily 0). In other words, $M_\sqcup$ is the Y_Z in Part 1 of Section 3.6. Thus $M_\# \simeq Y_0$, and $[w_\#] = c_1(\mathfrak{s}_\#)$ is nontorsion. Choose the metric on Y_0 to be from the set *Met* from Proposition 3.8.

In both cases above, M_1 is of the type Y_Z in Section 3.2; and hence contains a special 1-cycle γ. We denote this by γ_1. Consequently, assuming that p_1 is disjoint from γ_1, both $M_\sqcup$ and $M_\#$ inherit a 1-cycle from $\gamma_1 \subset M_1$. They are respectively denoted by $\gamma_\sqcup$ and $\gamma_\#$. According to Section 3.8, the filtered monopole Floer homologies

$HM^\circ(M_1, \langle w_1\rangle; \Lambda_{\gamma_1})$ and $HM^\circ(M_\#, \langle w_\#\rangle; \Lambda_{\gamma_\#})$ are well-defined. In parallel to what was done in Section 2.6, define $CM^\circ(M_\sqcup, \langle w_\sqcup\rangle; \Lambda_{\gamma_\#})$ to be the product complex of $CM^\circ(M_1, \langle w_1\rangle; \Lambda_{\gamma_1})$ and $\hat{C}(M_2, rw_2)$. The map $U_\sqcup = \hat{U}_{M_\sqcup}$, as given in (2.18), acts on $CM^\infty(M_\sqcup)$ and maps $CM^-(M_\sqcup) \subset CM^\infty(M_\sqcup)$ into itself. The same notation is used to denote its induced maps on $CM^+(M_\sqcup)$ and $CM^-(M_\sqcup)$. By construction, the four flavors $CM^\circ(M_\sqcup)$ are related by short exact sequences of the form (3.16). Thus by Lemma 4.2, the $H_*(S^1)$-modules $S_{U_\sqcup}(CM^\circ(M_\sqcup))$ are related by short exact sequences of the same form. The long exact sequences induced are also called the fundamental exact sequences.

The remainder of this subsection consists of three parts. The first part contains a filtered analog of Proposition 6.2. The second part analyzes the filtered connected sum formula from Part 1. This last part derives Theorem 1.1 from this computation.

Part 1: A filtered variant of Proposition 6.2 states:

Proposition 6.4 *Let $M_\sqcup$, $M_\#$ be as in either cases of (6.17). Then there is a system of isomorphisms from $HM^\circ(M_\#, \langle w_\#\rangle; \Lambda_{\gamma_\#})$ to $H_*(S_{U_\sqcup}(CM^\circ(M_\sqcup))$, $\circ = -, \infty, +, \wedge$ as graded $\mathbf{A}_\dagger(M_\#) \simeq \mathbf{A}_\dagger(M_\sqcup)$-modules, which is natural with respect to the fundamental exact sequences on both sides.*

Proof. Both cases in (6.17) satisfy the conditions of Proposition 6.2 (a). Take $\Gamma_1 = \Lambda_{\gamma_1}$, $\Gamma_2 = \mathbb{K}$ (the constant local coefficients). Then $\Gamma_\sqcup = \Lambda_{\gamma_\sqcup}$ and $\Gamma_\# = \Lambda_{\gamma_\#}$. Repeat the proof of Proposition 6.2 using cobordisms (X, w_X) constructed from Proposition 3.12 for Case 1) of (6.17), and Proposition 3.10 for Case 2). Like in the previous section, we denote this by the shorthand $\mathcal{V}$ when $Y_- = M_\#$, and by $\overline{\mathcal{V}}$ when $Y_- = M_\sqcup$. By construction, there is a cylinder $C \subset X$ ending at $\gamma_\sqcup \subset Y_\sqcup$, and $\gamma_\# \subset Y_\#$ satisfying the constraints in Section 3.7. According to Section 3.8, this gives us chain maps

$$\begin{aligned} &m^\infty[u](X, \langle w_X\rangle; \Lambda_C)\colon\ CM^\infty(Y_-) \to CM^\infty(Y_+) \quad \text{and} \\ &m^-[u](X, \langle w_X\rangle; \Lambda_C)\colon\ CM^-(Y_-) \to CM^-(Y_+). \end{aligned}$$

In parallel to (6.5), let

$$\begin{aligned} &V_0^\circ = m^\circ[1](\mathcal{V}; \Lambda_C), \quad V_1^\circ = m^\circ[u_\lambda](\mathcal{V}; \Lambda_C), \\ &V_1^{\dagger,\circ} = m^\circ[u_\lambda^-](\overline{\mathcal{V}}; \Lambda_C), \quad V_0^{\dagger,\circ} = m^\circ[1](\overline{\mathcal{V}}; \Lambda_C) \end{aligned}$$

for $\circ = -, \infty$, and use them to define V_*°, $V_*^{\dagger,\circ}$ as in (6.4). Keeping in mind the non-negativity of the integers $n(\mathfrak{d})$ entering the definitions of ∂^∞ and m^∞, the rest of the proof of Proposition 6.2 may be repeated with only cosmetic changes to see

that V_*° and $V_*^{\dagger,\circ}$ induce chain homotopy equivalences between $CM^\circ(M_\#, \langle w_\# \rangle; \Lambda_{\gamma_\#})$ and $S_{U_\sqcup}(CM^\circ(M_\sqcup))$, for $\circ = -, \infty$ These fit into commutative diagrams with the fundamental exact sequences (3.16) on both sides of V_*°, $V_*^{\dagger,\circ}$. This understood, the rest of the proposition follows from the Five Lemma. □

Part 2: We next analyze the homologies $H_*(S_{U_\sqcup}(CM^\circ(M_\sqcup))$ in the two cases of (6.17) respectively.

Case 1): Choose a product metric with constant curvature on $M_2 = S^1 \times S^2$. The moduli space of Seiberg-Witten solutions over it is a circle of flat connections. Choose a real Morse function on this circle with a pair of index 1 and index 0 critical points, and two gradient flow lines between them. Perform a perturbation to the Seiberg-Witten equations adapted to this Morse function, as described in Chapter 33 of [KM]. In this context $\hat{C}(M_2) = C^u = \mathbb{K}[u_2, y_2]$, $\partial_{M_2} = 0$, where:

- the unit $1 \in \mathbb{K}[u_2, y_2]$ has grading $[\xi_+]$ in the notation of [KM], p.57.
- u_2 has degree -2, and the U_2-map acts by multiplication by u_2.
- y_2 has degree 1 and represents a generator of $H_1(S^1;\mathbb{Z})$ co-oriented with the moduli spaces. In particular, $y_2^2 = 0$.

Thus,

$$\begin{aligned} S_{U_\sqcup}(CM^\circ(M_\sqcup)) &= CM^\circ(M_1)[u_2, y_2] \otimes \mathbb{K}[y], \\ D_\sqcup &= \partial_{M_1} \otimes \jmath + (U_1 \otimes -1 \otimes u_2) \otimes y. \end{aligned} \tag{6.18}$$

Write a generic element $a \in S_{U_\sqcup}(CM^\circ(M_\sqcup)$ as

$$a_0 + a_1 y, \quad \text{where } a_0, a_1 \in CM^\circ(M_1)[u_2, y_2].$$

Then

$$D_\sqcup a = \partial_{M_1} a_0 - (\partial_{M_1} a_1) y + (U_1 - u_2)(a_0) y.$$

Thus,

$$\begin{aligned} &H_*(S_{U_\sqcup}(CM^\circ(M_\sqcup)) \\ &= \{\, a_0 + a_1 y \,|\, \partial_{M_1} a_0 = 0, (U_1 - u_2)\, a_0 = \partial_{M_1} a_1 \} \otimes \mathbb{K}[y] \qquad \text{mod} \\ &\qquad \big(\partial_{M_1} b_1 y \sim 0, u_2 b_0 y \sim U_1 b_0 y - \partial_{M_1} b_0\big) \otimes \mathbb{K}[y_2] \\ &\simeq HM^\circ(M_1)\, y \otimes \mathbb{K}[y_2]. \end{aligned}$$

Consequently,

$$H_*(S^\circ_{U_\sqcup}(M_\sqcup)) \simeq HM^\circ(M_1)[y_2].$$

(Alternatively, use a spectral sequence computation, filtrate (6.18) first by degree in y, then by degree in u_2).

Case 2): In this case, $\mathfrak{C}(M_1) = \mathfrak{C}^o(M_1)$ consists of a single irreducible point, $(A, (\alpha, \beta)) = (0, ((2\mathrm{r})^{-1/2}, 0))$. (See e.g. [D1] for this well-known fact). Thus, $CM^\circ(M_1)$ and the fundamental short exact sequences relating them are simply the modules V° and the sequences in (4.5), (4.6), (4.7). Write the variable u in (4.5) as u_1 below. As pointed out in Remark 3.15 (a), u_1 stands both for the deck transformation and U-map on $CM^\circ(M_1)$.

This said, we have in this case

$$\begin{aligned} S_{U_\sqcup}(CM^\circ(M_\sqcup)) &= V^\circ(u_1) \otimes CM(M, c_-) \otimes \mathbb{K}[y], \\ D_\sqcup &= 1 \otimes \partial_M \otimes \jmath + (u_1 \otimes 1 - 1 \otimes U_2) \otimes y. \end{aligned} \tag{6.19}$$

This can alternatively be written as

$$\begin{aligned} &E^\circ\big(CM(M, c_-) \otimes \mathbb{K}[y], \partial_M \otimes \jmath - U_2 \otimes y\big) \\ &\qquad = E^\circ\big(S_{U_2}(CM(M, c_-))\big). \end{aligned} \tag{6.20}$$

By Proposition 5.9 and Remark 5.10, the homology of the latter is $\mathring{HM}(M, c_b)$, and the isomorphisms from $H_*(S_{U_\sqcup}(CM^\circ(M_\sqcup)))$ to the latter preserves the $\mathbb{K}[u]$-module structure and are natural with respect to the fundamental exact sequences. Since the U-map commutes with the $\bigwedge^* H_1(M;\mathbb{Z})/\,\mathrm{Tors}$-actions on both sides, These are isomorphisms as $\mathbf{A}_\dagger(M)$-modules.

To conclude, combining the above computation with Propostion 6.4, we have:

Corollary 6.5 *1) There is a system of isomorphisms of $\mathbf{A}_\dagger(M)$-modules*

$$HM^\circ(Y_i, \langle w\rangle; \Lambda_\gamma) \simeq HM^\circ(Y_{i-1}, \langle w\rangle; \Lambda_\gamma) \otimes H_*(S^1) \quad \textit{for } i = 1, \dots, \mathrm{G}$$

preserving the relative gradings and natural with respect to the fundamental exact sequences.

2) There is a system of isomorphisms of $\mathbf{A}_\dagger(M)$-modules

$$HM^\circ(Y_0, \langle w\rangle; \Lambda_\gamma) \simeq \mathring{HM}(M, c_b)$$

preserving the relative gradings and natural with respect to the fundamental exact sequences, respectively for $\circ = -, \infty, +, \wedge$ on the left hand side, and $\circ = \wedge, -, \vee, \sim$ on the right hand side.

Proof of Theorem 1.1: (1): This follows from an iteration of Corollary 6.5 1) and Lemma 6.6 below, in terms of the alternative notation (3.17).

Lemma 6.6 *There is a system of isomorphisms of $\mathbf{A}_\dagger(Y)$-modules*

$$HM^\circ(Y, \langle w\rangle; \Lambda_\gamma) \xrightarrow{\simeq} HM^\circ(Y_{\mathrm{G}}, \langle w\rangle; \Lambda_\gamma)$$

preserving the relative gradings and natural with respect to the fundamental exact sequences.

Proof. Y and Y_{G} stand for the same manifold with different metrics and associated 2-form w. As mentioned in Section 2.6, chain homotopies between the corresponding monopole Floer complexes are provided by chain maps induced from cobordisms $X = \mathbb{R} \times Y$ equipped with metrics and self-dual 2-forms interpolating those associated to Y_- and Y_+. (See e.g. Section IV.7.c for this type of argument). In our setting, choose X with the metrics and self-dual 2-forms over it to be those constructed in 3.13. This construction also provides a cylinder $C \subset X$ ending at Y's and Y_{G}'s version of γ. which induces X-morphisms Λ_C between Y's and Y_{G}'s version of Γ_γ. The positivity result in Proposition 3.4 guarantees that these chain maps are filtration-preserving, namely they map Y_-'s version of $CM^- \subset CM^\infty$ to Y_+'s version of $CM^- \subset CM^\infty$. As in the end of the proof of Proposition 6.4, their induced maps on homology together with the Five Lemma supply the isomorphisms asserted in the lemma. □

(2): This is a re-statement of Corollary 6.5 2) in alternative notation according to the second bullet of (3.17). □

7 Properties of solutions to (2.4)

This section supplies proofs for Lemma 3.1 and Proposition 3.6. Even so, much of what is done here is either used in Section 8 or has analogs in Section 8. Section 7.3 has the proof of Lemma 3.1 and Section 7.8 has the proof of Proposition 3.6.

By way of a convention, the manifold Z is assumed implicitly to be connected except in Section 7.8's proof of Proposition 3.6.

7.1 Pointwise bounds

Fix a Riemannian metric on Y_Z and a closed 2-form, denoted by w, whose de Rham class is that of $c_1(\det(\mathbb{S}))$. The four parts of this subsection assume such data so as to supply a priori pointwise bounds for the $C^\infty(Y_Z; \mathbb{S})$-component of any given pair in $\mathrm{Conn}(E) \times C^\infty(Y_Z; \mathbb{S})$ that obeys (2.4).

Part 1: The first lemma asserts relatively crude bounds which are subsequently refined.

Lemma 7.1 *There exists $\kappa > \pi$ with the following significance: Fix $\mathrm{r} \geq \kappa$ and an element $\mu \in \Omega$ with $\mathcal{P}$-norm less than 1. Let (A, ψ) denote a solution to the (r, μ)-version of (2.4). Then $|\psi| + \mathrm{r}^{-1/2}|\nabla_A\psi| + \mathrm{r}^{-1}|\nabla_A\nabla_A\psi| \leq \kappa\,(\sup_{Y_Z}|w|^{1/2} + \mathrm{r}^{-1/2})$.*

Proof. If w is identically zero, write $\psi = \mathrm{r}^{-1/2}\lambda$. The pair (A, λ) obeys the $\mathrm{r} = 1$ version of (2.4). In this case, the standard differntial equation techniques give the desired bounds. See for example what is said in Chapter 5 of [KM].

Granted the case when w vanishes identically, assume for what follows that $w \neq 0$ at points on Y_Z. The bound on $|\psi|$ follows by first using the Weitzenböck formula for the Dirac operator to see that $|\psi|^2$ obeys a differential inequality that has the schematic form:

$$d^\dagger d|\psi|^2 + 2|\nabla_A\psi|^2 + 2\mathrm{r}(|\psi|^2 - |w| - c_0\mathrm{r}^{-1})|\psi|^2 \leq 0. \tag{7.1}$$

Use the maximum principle with (7.1) to see that $|\psi|^2 < c_0 \sup_{Y_Z}|w|$. To see about the norm of $|\nabla_A\psi|$, fix a point in Y_Z and $\rho > 0$. Use χ to construct a function that equals 1 on the ball of radius ρ about the point and vanishes on the ball of radius 2ρ. Multiply both sides of (7.1) by this function and integrate. Use an integration by parts and the bound on $|\psi|^2$ to see that the integral of $|\nabla_A\psi|^2$ over the ball in Y_Z is at most $c_0(\rho + \mathrm{r}\rho^3)$. This holds for all ρ and all balls. In particular, it implies that the integral of $|\nabla_A\psi|^2$ on any ball in Y_Z of radius $\mathrm{r}^{-1/2}$ is bounded by $c_0\mathrm{r}^{-1/2}$. Hold on to this last bound for the moment.

Differentiate the equation $D_A^2\psi = 0$, commute covariant derivatives and use the Weitzenböck formula again to obtain a differential inequality for $|\nabla_A\psi|^2$ that reads $d^\dagger d|\nabla_A\psi|^2 \leq c_0(\mathrm{r}|\nabla_A\psi|^2 + 1)$. The Dirichlet Green's function for balls of radius $c_0^{-1}\mathrm{r}^{-1/2}$ can be used with this last inequality and the bound on the integral of $|\nabla_A\psi|^2$ over balls of radius $\mathrm{r}^{-1/2}$ to obtain the asserted bound for $|\nabla_A\psi|$ at the center of any ball of this radius. Differentiate the equation $D_A^2\psi = 0$ twice to obtain an equation for $\nabla_A\nabla_A\psi$. Granted the bounds just derived for $|\psi|$ and $|\nabla_A\psi|$, then the same sort of Green's function argument on balls of radius $c_0^{-1}\mathrm{r}^{-1/2}$ will give the asserted bound for $|\nabla_A\nabla_A\psi|$. □

Part 2: This part of the subsection sets the notation for what is to come in Part 3 and in the subsequent sections. To start, introduce K_*^{-1} to denote the 2-plane subbundle of

the tangent bundle over the $|w| > 0$ part of Y_Z given by the kernel of $*w$. Orient K_*^{-1} by the restriction of w and use the induced metric with this orientation to view K_*^{-1} as a complex line bundle. Clifford multiplication by the 1-form $*w$ on the $|w| > 0$ part of Y_Z writes $\mathbb{S}$ as a direct sum of eigenbundles $E_* \oplus (E_* \otimes K_*^{-1})$ with E_* being the $+i|w|$ eigenbundle.

Use $\mathbb{I}_{\mathbb{C}}$ to denote the product complex line bundle and θ_0 to denote the product connection on $\mathbb{I}_{\mathbb{C}}$. Let $1_{\mathbb{C}}$ denote the θ_0 -constant section of $\mathbb{I}_{\mathbb{C}}$ with value 1 at all points. Fix a unitary identification between $E_*^{-1} \otimes_{\mathbb{C}} E_*$ and $\mathbb{I}_{\mathbb{C}}$ and use the latter to write $E_*^{-1} \otimes_{\mathbb{C}} \mathbb{S}$ as $\mathbb{I}_{\mathbb{C}} \oplus K_*^{-1}$. The bundle K_*^{-1} has a canonical connection, which we denote by A_{K_*}, such that the section $(1_{\mathbb{C}}, 0)$ of the bundle $\mathbb{I}_{\mathbb{C}} \oplus K_*^{-1}$ obeys the Dirac equation as defined using the connection $A_{K_*} + 2\theta_0$ on its determinant line bundle. The norm of the curvature of A_{K_*} is bounded by $c_0|w|^{-2}$ and the norm of the k-th derivative of A_{K_*}'s curvature is bounded by $c_k|w|^{-2-k}$ with c_k being a constant.

A section ψ of $\mathbb{S}$ over U is written with respect to this splitting as $|w|^{1/2}(\alpha, \beta)$. Meanwhile, the connection A on E defines a corresponding connection on E_*, that is, the connection $A_* = A - \frac{1}{2}(A_K - A_{K_*})$. To keep the notation under control in what follows, the A_*-covariant derivative on E_* is also denoted by ∇_A, as is the $A_* + A_{K_*}$-covariant derivative on $E_* \otimes K_*^{-1}$.

Part 3: The next lemma refines Lemma 7.1's bound on the $|w| > 0$ part of Y_Z.

Lemma 7.2 *There exists $\kappa > \pi$ with the following significance: Fix $\mathrm{r} \geq \kappa$ and an element $\mu \in \Omega$ with $\mathcal{P}$-norm less than 1. Let (A, ψ) denote a solution to the (r, μ)-version of (2.4). Fix $m \in (\kappa, \kappa \mathrm{r}^{1/3}(\ln \mathrm{r})^{-\kappa})$ and let U_m denote the $|w| > m^{-1}$ part of Y_Z. Write ψ on U_m as $|w|^{1/2}(\alpha, \beta)$. Then the pair (α, β) obeys the following on U_m:*

- $|\alpha|^2 \leq 1 + \kappa m^3 \mathrm{r}^{-1}$.
- $|\beta|^2 \leq \kappa m^3 \mathrm{r}^{-1}(1 - |\alpha|^2) + \kappa^3 m^6 \mathrm{r}^{-2}$.
- $|\nabla_A \alpha|^2 + m^{-3}\mathrm{r}|\nabla_A \beta|^2 \leq \kappa m^{-1}\mathrm{r}(1 - |\alpha|^2) + \kappa^2 m^2$.
- *Denote by U_* the $1 - |\alpha|^2 \geq \kappa^{-1}$ part of U_m. Then*

$$|1 - |\alpha|^2| \leq (m^2 e^{-\sqrt{\mathrm{r}/m}\, \mathrm{dist}(\cdot, U_*)/\kappa} + \kappa m^3 \mathrm{r}^{-1}).$$

Proof. Write $\psi = |w|^{1/2}\eta$ on U_{2m}. The section η on U_{2m} obeys an equation having the schematic form $D_A \eta + \mathfrak{R} \cdot \eta = 0$ with $\mathfrak{R}$ being Clifford multiplication by the 1-form $\frac{1}{2}d(\ln |w|)$. Note in particular that $|\mathfrak{R}| \leq c_0\, m$ and the absolute value of the

covariant derivative of $\mathfrak{R}$ is bounded by $c_0\, m^2$. Use the Weitzenböck formula for the operator $D_A + \mathfrak{R}$ to see that η obeys an equation that has the schematic form

$$\nabla_A^\dagger \nabla_A \eta - \mathrm{cl}(B_A) \cdot \eta + \mathfrak{R}_1 \cdot \nabla_A \eta + \mathfrak{R}_0 \cdot \eta = 0, \tag{7.2}$$

where $\mathrm{cl}(\cdot)$ denotes the Clifford multiplication endomorphism from T^*M to $\mathrm{End}(\mathbb{S})$ and where $\mathfrak{R}_1$ and $\mathfrak{R}_0$ are linear and obey $|\mathfrak{R}_1| \le c_0 m$ and $|\mathfrak{R}_0| \le c_0 m^2$. Let q denote the maximum of 0 and $|\eta|^2 - 1 - c_0 m^3 \mathrm{r}^{-1}$. It follows from (7.2) that q on U_{2m} obeys

$$d^\dagger d q + 2\mathrm{r} m^{-1} q \le 0. \tag{7.3}$$

As Lemma 7.1 bounds q by $c_0 m$ on U_{2m}, the comparison principle with the Green's function for the operator $d^\dagger d + \mathrm{r} m^{-1}$ to see that $q \le c_0 m^3 \mathrm{r}^{-1}$ on $U_{3m/2}$. This implies the claim in the first bullet. It also implies that $|\beta|^2$ is less than $1 + c_0 m^3 \mathrm{r}^{-1}$ on $U_{3m/2}$.

To see about the second bullet, project (7.2) onto the $E_* \otimes K_*^{-1}$ summand of $\mathbb{S}$ and take the fiberwise inner product of the resulting equation with β to obtain a differential inequality that has the form

$$d^\dagger d |\beta|^2 + 2\mathrm{r} m^{-1} |\beta|^2 \le -|\nabla_A \beta|^2 + c_0 \mathrm{r}^{-1} m^3 |\nabla_A \alpha|^2 + c_0 \mathrm{r}^{-1} m^5. \tag{7.4}$$

Fix for the moment $\varepsilon > 0$. Project (7.2) next onto the E_* summand and take the pointwise inner product with α to obtain an equation for the function $\mathrm{w} = 1 - |\alpha|^2$ that has the form

$$d^\dagger d \mathrm{w} + 2\mathrm{r} m^{-1} \mathrm{w} = 2|\nabla_A \alpha|^2 + \mathrm{r} m^{-1} \mathrm{w}^2 + \mathfrak{e}, \tag{7.5}$$

where $|\mathfrak{e}| \le c_0 \varepsilon |\nabla_A \beta|^2 + c_0(1 + \varepsilon^{-1}) m^2 + c_0 m |\nabla_A \alpha|$.

It follows from (7.4) and (7.5) that there exist constants z_1 and z_2 that are both bounded by c_0 and $\varepsilon > c_0^{-1}$ such that the function $q = |\beta|^2 - z_1 \mathrm{r}^{-1} m^3 \mathrm{w} - z_2 \mathrm{r}^{-2} m^6$ obeys the equation

$$d^\dagger d q + 2\mathrm{r} m^{-1} q \le 0 \tag{7.6}$$

on $U_{3m/2}$. Granted this inequality, use the Green's function for $d^\dagger d + \mathrm{r} m^{-1}$ as before to see that $|\beta|^2 \le z_1 m^3 \mathrm{r}^{-1} (1 - |\alpha|^2) + z_2 m^6 \mathrm{r}^{-2}$ on U_m.

The proofs of the third and fourth bullets start by differentiating (7.2) to obtain an equation for the components of $\nabla_A \eta$ and it then copies the manipulations done in Step 2 of Section 4d in [Ta1] to obtain a differential inequality on $U_{3m/2}$ for the function $\mathfrak{h} := |\nabla_A \eta|^2$ that has the form

$$d^\dagger d \mathfrak{h} + 2\mathrm{r} m^{-1} \mathfrak{h} \le c_0 (\mathrm{r} m^{-1} \mathrm{w} \mathfrak{h} + m^2 \mathfrak{h} + m^4 + \mathrm{r}^2 m^{-2} \mathrm{w}^2). \tag{7.7}$$

To prove the third bullet, use (7.4), (7.5) and (7.7) to find constants z_1, $z_2 > 0$ and z_3, all with absolute value less than c_0, such that the function $q := \mathfrak{h} - z_1 \mathrm{r} m^{-1}(1-|\alpha|^2) - z_2 m^2 + z_3 \mathrm{r} m^{-1}|\beta|^2$ obeys (7.6) on U_{2m} when $m < c_0^{-1}\mathrm{r}^{1/3}$. Meanwhile, Lemma 7.1 implies that $\mathfrak{h}$ is no larger than $c_0 m\mathrm{r}$ on U_{2m}. Given this last bound, the comparison argument that uses the Green's function for $d^\dagger d + c_0^{-1}\mathrm{r}m^{-1}$ says that $|\nabla_A\eta|^2$ is bounded by $c_0 m^{-1}\mathrm{r}(1-|\alpha|^2) + c_0^2 m^2$ on U_m when $m \le c_0^{-1}\mathrm{r}^{1/3}$. This gives Lemma 7.2's bound for $|\nabla_A\alpha|^2$. The refinement that gives the asserted bound for $|\nabla_A\beta|^2$ is obtained by the same sort of argument after first projecting (7.2) onto the $E_* \otimes K_*^{-1}$-summand of $\mathbb{S}$ before differentiating so as to get an elliptic equation for $\nabla_A\beta$. The details of this part of the story are straightforward and omitted.

To prove the fourth bullet, use the first bullet of the lemma with (7.5) and (7.7) to see that $q := \mathfrak{h} + c_0^{-1}\mathrm{r}m^{-1}\mathrm{w} - c_0 m^2$ obeys an equation on the $\mathrm{w} \le c_0^{-1}$ part of U_{2m} that has the form $d^\dagger dq + c_0^{-1}\mathrm{r}m^{-1}q \le 0$ when $m \le c_0^{-1}\mathrm{r}^{1/3}$. Granted the latter and granted the a priori bound $q \le c_0\mathrm{r}m$ from Lemma 7.1, then the comparison principle using the Green's function for $d^\dagger d + c_0^{-1}\mathrm{r}m^{-1}$ leads to the following: If $c > c_0$, then $q \le c_0 \mathrm{r}m\, e^{-\sqrt{\mathrm{r}/m}\,\mathrm{dist}(\cdot,U_c)/c_0}$ where U_c denotes the $\mathrm{w} \ge c^{-1}$ part of U_{2m}. This last inequality implies Lemma 7.2's fourth bullet. □

Part 4: The final lemma of this subsection refines what is said by Lemma 7.1 on the part of Y_Z where $|w|$ is positive but small.

Lemma 7.3 *There exists $\kappa > 1$ with the following property: Fix $m \in (\kappa, \kappa^{-1}\mathrm{r}^{1/3}(\ln \mathrm{r})^{-\kappa})$. Fix $\mathrm{r} \ge \kappa$ and fix $\mu \in \Omega$ with $\mathcal{P}$-norm less than 1 and let (A,ψ) be a solution to the (r,μ)-version of (2.4). Then $|\psi| \le \kappa m^{-1/2}$ and $|\nabla_A\psi| \le \kappa m^{-1}\mathrm{r}^{1/2}$ on the $|w| < m$ part of Y_Z.*

Proof. The maximum principle as applied to (7.1) implies that $|\psi|^2$ can not have a local maximum where $|\psi|^2 > |w| + c_0\mathrm{r}^{-1}$. As Lemma 7.2 finds $|\psi|^2 \le c_0 m^{-1}$ on the boundary of U_m, the maximum principle implies that $|\psi|^2 \le c_0 m^{-1}$ on where $|w| < m^{-1}$. To see about $|\nabla_A\psi|$, let $p \in Y_Z$ denote a given point where $w \le 2m^{-1}$. Fix Gaussian coordinates for a ball of radius c_0^{-1} centered at p and then rescale the coordinates so that the ball of radius $m^{-1/2}\mathrm{r}^{1/2}$ about the origin in $\mathbb{R}^3$ and radius 1. Let φ denote the corresponding map from the ball of radius 1 about the origin in $\mathbb{R}^3$ to the original ball in Y_Z. With this understood, the pull-back $(\varphi^*A, m^{1/2}\varphi^*\psi)$ satisfies a version of (2.4) on the unit ball in $\mathbb{R}^3$ that is defined by the rescaled metric. It follows from the bound on $|\psi|$ that $|B_A| \le c_0 m^{-1}\mathrm{r}$ and this implies that $|\varphi^* B_A| \le c_0$. This understood, standard elliptic regularity techniques can be employed to see that the

rescaled version of $m^{1/2}|\varphi^*(\nabla_A\psi)|$ has norm bounded by c_0 and so $|\nabla_A\psi|$ has norm bounded by $c_0 m^{-1}\mathrm{r}^{1/2}$. $\square$

7.2 The micro-local structure of (A, ψ)

Part 3 of this section states and then proves Lemma 7.4, this being a lemma that describes solutions to (2.4) on the $|w| > 0$ part of Y_Z when viewed with microscope that magnifies by a factor of the order of $\mathrm{r}^{1/2}$. Parts 1-2 of the subsection set the notation that is used in particular for Lemma 7.4 but elsewhere as well.

Part 1: This part of the subsection introduces the *vortex equations* on $\mathbb{C}$. This is a system of equations that asks that a pair (A_0, α_0) of connection on a complex line bundle over $\mathbb{C}$ and section of this bundle obey

$$(7.8)\qquad \begin{cases} *F_{A_0} = -i(1 - |\alpha_0|^2), \\ \bar{\partial}_{A_0}\bar{\alpha}_0 = 0, \\ |\alpha_0| \leq 1. \end{cases}$$

The notation here is such that $*$ denotes the Euclidean Hodge dual on $\mathbb{C}$, while F_{A_0} and $\bar{\partial}_{A_0}$ denote the respective curvature 2-form of A_0 and the d-bar operator defined by A_0 on the space of sections of the given complex line bundle. Note that if (A_0, α_0) is a solution to (7.8), then so is $(A_0 - u^{-1}du, u\,\alpha_0)$ with u being any smooth map from $\mathbb{C}$ to S^1.

Solutions with $1 - |\alpha_0|^2$ integrable are discussed at length in Sections 1 and 2 of [T2], Section IV.2b and Section IV.3a. As noted in these references, if $1 - |\alpha_0|^2$ is integrable then its integral is 2π times a non-negative integer. Fix $\mathrm{m} \in \{0, 1, \ldots\}$. The space of $C^\infty(\mathbb{C}; S^1)$ equivalence classes of solutions to (7.8) with the integral of $1 - |\alpha_0|^2$ equal to $2\pi\mathrm{m}$ has the structure of a smooth, $2\mathrm{m}$-dimensional manifold. This manifold is denoted in what follows by $\mathfrak{C}_\mathrm{m}$. By way of a parenthetical remark, the space $\mathfrak{C}_\mathrm{m}$ has a natural complex structure that identifies it with $\mathbb{C}^\mathrm{m}$. A solution with $1 - |\alpha_0|^2$ integrable is said here to be a *finite energy solution* to the vortex equation.

Part 2: Lemma 7.4 and some of the later subsections refer to the notion of a transverse disk with a given radius through a given $|w| > 0$ point in Y_Z. A *transverse disk* is the image via the metric's exponential map of the centered disk of the given radius in the 2-plane bundle $\mathrm{Ker}(*w)$ at the given point. There exists $c_0 > 100$ such that any

transverse disk with radius c_0^{-1} is embedded with a priori bounds on the derivatives to any given order of its extrinsic curvature. If $D \subset Y_Z$ is a transverse disk centered at a point p, and if $c \geq c_0$, then $|w|$ will be greater than $\frac{1}{2}|w|(p)$ on the subdisk in D centered at p with radius $c^{-1}|w|(p)$. The constant c can be chosen so that the following is also true: Let v denote the vector field on the $|w| > 0$ part of Y_Z that generates the kernel of w and has pairing 1 with $*w$. Then v is orthogonal to D at p and the length of the projection to TD of v on the concentric disk in D of radius $c^{-1}|w|(p)$ is no greater than $c_0 c^{-1}$. Choose $c \geq c_0$ with this property and use D_p to denote the transverse disk through p of radius $c^{-1}|w|(p)$.

Reintroduce from Part 2 of Section 7.1 the complex line bundle K_*^{-1} defined over the $|w| > 0$ part of Y_Z. Recall that the underlying real bundle is the 2-plane bundle in TY_Z annihilated by $*w$. Let p again denote a point in the $|w| > 0$ part of Y_Z. Fix an isometric isomorphism from $K_*^{-1}|_p$ to $\mathbb{C}$. Use φ in what follows to denote the map from $\mathbb{C}$ to Y_Z that is obtained by composing first the isomorphism with $K_*|_p = \mathrm{Ker}(*w)|_p$ and then the metric's exponential map. With $\mathrm{r} \geq 1$ given, use φ_r to denote the composition of first multiplication by $\mathrm{r}^{-1/2}|w(p)|^{-1/2}$ on $\mathbb{C}$ and then applying φ.

To finish the notational preliminaries, suppose that (A, ψ) is a given pair in $\mathrm{Conn}(E) \times C^\infty(Y_Z; \mathbb{S})$. Write ψ where $|w| > 0$ as $|w|^{1/2}(\alpha, \beta)$ to conform with Part 2 of Section 7.1's splitting of $\mathbb{S}$ as $E_* \oplus (E_* \otimes K_*^{-1})$. Likewise reintroduce from Part 2 of Section 7.1 the connection A_* on the bundle E_*. Given $p \in Y_Z$ with $|w(p)| > 0$, introduce $(A_\mathrm{r}, \alpha_\mathrm{r})$ to denote the φ_r-pull back of the pair (A_*, α) to the radius $c^{-1}\mathrm{r}^{1/2}|w(p)|^{1/2}$ disk in $\mathbb{C}$.

Part 3: Lemma 7.4 below characterizes the pair $(A_\mathrm{r}, \psi_\mathrm{r})$.

Lemma 7.4 *There exists $\kappa > 10$ and given $\mathrm{R} > \kappa$, there exists $\kappa_\mathrm{R} > 1$ with the following property: Fix $\mathrm{r} \geq \kappa_\mathrm{R}$ and $\mu \in \Omega$ with $\mathcal{P}$-norm bounded by 1 . Suppose that (A, ψ) is a solution to the (r, μ)-version of (2.4). Fix a point in Y_Z where $|w| > \mathrm{r}^{-1/3}(\ln \mathrm{r})^\kappa$ and use the corresponding version of φ_r to obtain the pair $(A_\mathrm{r}, \alpha_\mathrm{r})$ of connection and section of a complex line bundle over $\mathbb{C}$. There exists a solution to the vortex equation on $\mathbb{C}$ whose restriction to the radius R disk about the origin in $\mathbb{C}$ has C^1-distance less than R^{-4} from $(A_\mathrm{r}, \alpha_\mathrm{r})$ on this same disk. Moreover, if $1 - |\alpha_\mathrm{r}|^2 < \frac{1}{2}$ at distances between $\mathrm{R} + \kappa(\ln \mathrm{R})^2$ and $\mathrm{R} - \kappa(\ln \mathrm{R})^2$ from the origin, then $(A_\mathrm{r}, \alpha_\mathrm{r})$ has C^1-distance less than R^{-4} in the radius R-disk about the origin in $\mathbb{C}$ from a finite energy solution to the vortex equations that defines a point in some $\mathrm{m} \leq \pi \mathrm{R}^2$ version of $\mathfrak{C}_\mathrm{m}$.*

Proof. It follows from (2.4) and what is said by the first three bullets of Lemma 7.2 that the curvature of A_{r} and α_{r} are such that

$$*F_{A_{\mathrm{r}}} = -i(1-|\alpha_{\mathrm{r}}|^2) + \mathfrak{e}_0 \quad \text{and} \quad \bar{\partial}_{A_{\mathrm{r}}}\alpha_{\mathrm{r}} = \mathfrak{e}_1, \tag{7.9}$$

where $|\mathfrak{e}_0| + |\mathfrak{e}_1| \leq c_0(\ln \mathrm{r})^{-c_0}$ on the disk in $\mathbb{C}$ of radius less than $c^{-1}\mathrm{r}^{1/2}m^{-1/2}$. The third bullet in Lemma 7.2 also finds $|\nabla_{A_{\mathrm{r}}}\alpha_{\mathrm{r}}| \leq c_0$. Granted (7.9), then the argument used to prove Lemma 6.1 in [T:W1] can be used with only minor modifications to prove the assertion with C^1-distance replaced by the distance as measured by any $\upsilon < 1-\mathrm{R}^{-1}$ Hölder norm. The convergence in the C^1-topology follows using the arguments from Section 6 in [T:W1] given also the second derivative bound from Lemma 7.1. □

7.3 Holomorphic domains

What follows directly sets the notation for what is to come in this subsection. An open set $U \subset Y_Z$ is said to be a *holomorphic domain* when the following criteria are met:

- The metric has non-negative Ricci curvature on U.
- The 2-form w is non-zero on U and covariantly constant.
- The curvature of A_K on U is a multiple of w.
- The 1-form μ on U and its derivatives to order 10 have norm less than $e^{-\mathrm{r}^2/2}$.

The following lemma strengthens the conclusions of Lemma 7.2 on a holomorphic domain.

Lemma 7.5 *Let $U \subset Y_Z$ denote a holomorphic domain and let $U_1 \subset U$ denote an open set with compact closure in U. Use D to denote the function on U that measures the distance to $Y_Z - U$. There exists $\kappa > \pi$ with the following significance: Fix $\mathrm{r} \geq \kappa$ and a 1-form $\mu \in \Omega$ with $\mathcal{P}$-norm less than 1 whose norm on U and those of its first 10 derivatives is bounded by $e^{-\mathrm{r}^2/2}$. Suppose that (A, ψ) is a solution to the (r, μ)-version of (2.4). Write ψ on U as $|w|^{1/2}(\alpha, \beta)$. Then β on U_1 obeys:*

- $|\beta| \leq \kappa e^{-\sqrt{\mathrm{r}}\mathrm{D}/\kappa}$.
- *Given $q \geq 1$, there exists $\kappa_q \geq 1$ such that $|(\nabla_A)^q\beta| \leq \kappa_q e^{-\sqrt{\mathrm{r}}\mathrm{D}/\kappa}$ with κ_q depending only on the metric, A_K, U and U_1.*

Proof. The proof that follows assumes that $\mu = 0$ on U. The proof in the general case differs little from what is said below and is left to the reader.

Keep in mind that the norm of $|w|$ is constant on U because w is covariantly constant. Project the Weitzenböck formula for D_A^2 onto the $E_* \otimes K_*^{-1}$ summand of $\mathbb{S}$ to obtain an equation for β on U that has the schematic form:

$$\nabla_A^\dagger \nabla_A \beta + \mathrm{r}|w|(1 + |\alpha|^2 + |\beta|^2)\beta + \mathfrak{R}\beta = 0, \tag{7.10}$$

with $\mathfrak{R}$ determined solely by the metric and A_K. Granted this, then by the conditions on the metric and A_K over U, $|\beta|$ obeys an equation of the form $d^\dagger d|\beta| + \mathrm{r}|w||\beta| \le 0$ on U when r is larger than a constant that depends only on U and U_1. The bound in the first bullet of the lemma follows from the latter equation using the comparison principle and the Green's function for the operator $d^\dagger d + \mathrm{r}|w|$. Given the bounds from Lemma 7.2, very much the same strategy leads to the bounds in the subsequent bullets after differentiating (7.1) to obtain an equation for $(\nabla_A)^q\beta$. □

Lemma 7.5 leads directly to the next lemma that describes ψ on U_γ and $\mathcal{H}_0$.

Lemma 7.6 *Given $\varepsilon > 0$, there exists $\kappa \ge \pi$ with the following significance: Introduce U to denote $U_\gamma \cup \mathcal{H}_0$ and let D denote the function on U that measures the distance to $Y_Z - U$. Introduce $U_\varepsilon \subset U$ to denote the subset with $\mathrm{D} > \varepsilon$. Fix $\mathrm{r} \ge \kappa$ and a 1-form $\mu \in \Omega$ with $\mathcal{P}$-norm less than 1 whose norm on U and those of its first ten derivatives is bounded by $e^{-\mathrm{r}^2/2}$. Let (A, ψ) denote a solution to the (r, μ)-version of (2.4). The following is true on U_ε:*

- *The conclusions of Lemma 7.5 hold with U_1 therein set to U_ε.*
- $-\kappa e^{-\sqrt{\mathrm{r}}\mathrm{D}/\kappa} \le 1 - |\alpha|^2 \le \kappa e^{-\sqrt{\mathrm{r}}\mathrm{D}/\kappa}$.
- *Given $q \ge 1$, there exists $\kappa_q \ge 1$ such that $|(\nabla_A)^q\alpha| \le \kappa_q e^{-\sqrt{\mathrm{r}}\mathrm{D}/\kappa}$ with κ_q depending only on the metric, A_K, U and ε .*

Proof. The first bullet follows by virtue of the fact that $U_\gamma \cap \mathcal{H}_0$ is a holomorphic domain when the constraints in (3.5) and (3.6) are obeyed. To see about the other bullets of the lemma, suppose for the moment that $\delta > 0$, that $p \in \mathcal{H}_0 \cap U_\varepsilon$ and that $|\alpha| > \delta$ at p. It follows from the second bullet of Lemma 7.2 and Lemma 7.5 that the integral of $*B_A$ on the radius $c_0^{-1}\mathrm{r}^{-1/2}\delta$ disk in the constant u slice of $\mathcal{H}_0$ through p is greater than $c_0^{-1}\delta^3$. Lemma 7.5 implies that the pull-back of $*B_A$ to the constant u sphere through p can be written as $\frac{i}{4\pi}\mathrm{F}\sin\theta d\theta \wedge d\phi$ and that $\mathrm{F} \ge -c_0 e^{-\sqrt{\mathrm{r}}/c_0}$, and so the integral of $*B_A$ on this transverse sphere in $\mathcal{H}_0$ will be positive if $\delta > c_0 e^{-\sqrt{\mathrm{r}}/c_0}$. But this is not allowed by virtue of the assumption that E's first Chern class has zero pairing with the $H_2(\mathcal{H}_0; \mathbb{Z})$-summand in (3.4). This being the case, the second bullet follows from the fourth bullet of Lemma 7.2 and Lemma 7.5.

Suppose that $p \in U_\gamma \cap U_\varepsilon$. The Dirac equation writes the $\frac{\partial}{\partial t}$-covariant derivative of α as a linear combination of covariant derivatives of β. This understood, Lemma 7.5 implies that the absolute value of the $\frac{\partial}{\partial t}$-covariant derivative of α in U_γ is bounded by $c_0 e^{-\sqrt{\mathrm{r}}/c_0}$. It follows as a consequence that if $|\alpha| > \delta$ at a point in $U_\gamma \cap U_\varepsilon$, then $|\alpha| > \frac{1}{2}\delta$ at points in $\mathcal{H}_0 \cap U_\varepsilon$ and, as just explained, this is not allowed if $\mathrm{r} \geq c_0$ and $\delta > c_0 e^{-\sqrt{\mathrm{r}}/c_0}$. This being the case, the bound in the second bullet for points in $U_\gamma \cap U_\varepsilon$ follows from Lemma 7.5 and the fourth bullet of Lemma 7.2.

The assertion in the third bullet is proved by writing $\psi = |w|^{1/2}\eta$ on U. Keeping in mind that $|w|$ is constant on U, project the Weitzenböck formula for $D_A^2\psi$ onto the E-summand of $\mathbb{S}$ and differentiating to obtain an equation for $(\nabla_A)^q\alpha$. Given the first bullet of Lemma 7.6 and given Lemma 7.5, the latter implies a differential inequality for the function $\sigma := |(\nabla_A)^q\alpha|$ of the form $d^\dagger d\sigma + \mathrm{r}|w|\sigma \leq c_q e^{-\sqrt{\mathrm{r}}/c_0}$ when $q = 1$, and it implies an equality of this same sort for $q > 1$ if the second bullet holds for all $q' < q$. Here, c_q depends only on q. Use the Green's function for $d^\dagger d + \mathrm{r}|w|$ with this differential inequality for σ to prove the third bullet's assertion. □

Lemma 7.6 in turn leads to the

Proof of Lemma 3.1. If $\mathrm{r} \geq c_0$, then Lemma 7.6 asserts that $|\alpha|$ is very close to 1 on a neighborhood of γ and so what is denoted in (3.7) as $\wp(|\alpha|)$ is equal to 1 on this neighborhood. With this in mind, note that $\alpha|\alpha|^{-1}$ is $\hat{A}$-covariantly constant where $\wp = 1$. This implies that $\hat{A}$ has holonomy 1 along γ. Since A_E has holonomy 1 on γ, it follows that $\hat{A} - A_E$ on γ can be written as $i\,\hat{u}(t)\,dt$ with $\hat{u}$ being a function on $\mathbb{R}/(\ell_\gamma\mathbb{Z})$ whose integral is an integer multiple of 2π. □

7.4 The L^1-norm of B_A when w is harmonic

This section supplies a crucial bound for the integral of $|B_A|$ over Y_Z given an extra assumption about w.

Lemma 7.7 *Suppose that w is a harmonic 2-form and that the zeros of w are non-degenerate. There exists $\kappa \geq \pi$ with the following significance: Fix $\mathrm{r} \geq \kappa$ and a 1-form $\mu \in \Omega$ with $\mathcal{P}$-norm less than 1. Suppose that (A, ψ) is a solution to the (r, μ)-version of (2.4). Then $\int_{Y_Z} |w||B_A| \leq \kappa$ and $\int_{Y_Z} |B_A| \leq \kappa \mathrm{r}^{1/5}$.*

By way of a look ahead, the lemma's bound of $\kappa\mathrm{r}^{1/5}$ for the L^1-norm of B_A is replaced in Lemma 7.9 by the bound $(\ln \mathrm{r})^{c_0}$.

Proof. The proof has three steps. By way of an overview, the plan is to compare the integrals of $|B_A|$ and $|w|\,|B_A|$ with the integral of $w \wedge iB_A$. The point being that the absolute value of the latter integral enjoys an (A,ψ)-, r- and μ- independent bound by virtue of the fact that w is harmonic; it computes the cup product pairing between the de Rham class of $*w$ and 2π times the first Chern class of the bundle E.

Step 1: Fix $m \in (c_0, c_0\mathrm{r}^{1/3}(\ln \mathrm{r})^{-c_0})$ so as to invoke Lemmas 7.2 and 7.3. Use U_m to again denote the part of Y_Z where $|w| > m^{-1}$. Since w has non-degenerate zeros, the volume of $Y_Z - U_m$ is less than $c_0 m^{-3}$. Since $|B_A| \le c_0\mathrm{r}(|\psi|^2 + |w|) + c_0$, it follows from Lemma 7.3 that

$$\int_{Y_Z - U_m} |B_A| \le c_0 \mathrm{r} m^{-4} \quad \text{and} \quad \int_{Y_Z - U_m} |w \wedge B_A| \le c_0 \mathrm{r} m^{-5}. \tag{7.11}$$

Save these bounds for the moment.

Step 2: Fix $m \in (c_0, c_0\mathrm{r}^{1/3})$. Use the equations in (2.4) and Lemma 7.2 to see that $|B_A|$ on U_m obeys $|B_A| \le \mathrm{r}|w|(|1 - |\alpha|^2| + |\beta|) + c_0$. This understood, the first and second bullets in Lemma 7.2 imply that

$$|B_A| \le c_0 \mathrm{r}\,|w|(1 - |\alpha|^2) + c_0|w|m^3 \tag{7.12}$$

at all points in U_m. Meanwhile, use the equations in (2.4) to see that

$$w \wedge iB_A \ge \mathrm{r}\,|w|^2(1 - |\alpha|^2) - c_0|w| \tag{7.13}$$

on U_m. This lower bound and the upper bound in (7.12) imply that if $q \in \{0, 1\}$, then

$$|w|^q|B_A| \le c_0 m^{1-q}(w \wedge iB_A) + c_0 m^{2-q} \tag{7.14}$$

at all points in U_m.

Step 3: Fix for the moment $m_0 \ge c_0$ and a positive integer N with an upper bound such that $2^N m_0 < c_0^{-1}\mathrm{r}^{1/3}$. For $k \in \{1, 2, \ldots, N\}$, set $m_k := 2^k m_0$. Noting that the volume of $U_{m_k} - U_{m_{k-1}}$ is bounded by $c_0 2^{-3k}$, it follows from (7.14) that

$$\int_{U_{m_k} - U_{m_{k-1}}} |w|^q|B_A| \le c_0 m_N^{1-q} \int_{U_{m_k} - U_{m_{k-1}}} w \wedge iB_A + c_0 2^{-k}. \tag{7.15}$$

Sum the various $k \in \{1, \ldots, N\}$ versions of (7.15) to see that

$$\int_{U_{m_N}} |w|^q|B_A| \le c_0 m_N^{1-q} \int_{U_{m_N}} w \wedge iB_A + c_0. \tag{7.16}$$

This last inequality and the $m = m_N$ version of (7.11) imply that

$$\int_{Y_Z} |B_A| \leq c_0 m_N^{1-q} \int_{Y_Z} w \wedge iB_A + c_0(\mathrm{r} m_N^{-4-q} + 1). \tag{7.17}$$

The integral on the right hand side of (7.17) is in any event bounded by c_0 and so what is written in (7.17) leads to the bound

$$\int_{Y_Z} |w|^q |B_A| \leq c_0(m_N^{1-q} + \mathrm{r} m_N^{-4-q}). \tag{7.18}$$

This understood, take N so that $\mathrm{r}^{1/5} \leq m_N \leq c_0 \mathrm{r}^{1/5}$ to obtain Lemma 7.7's assertion. □

7.5 Where $1 - |\alpha|^2$ is not small

Suppose that (A, ψ) is a solution to a given (r, μ)-version of (2.4). Write ψ where $|w| > 0$ as $|w|^{1/2}(\alpha, \beta)$ and denote the version of κ that appears in Lemma 7.4 by $\kappa_\diamond$.

The lemma that follows momentarily characterizes the $|w| > \mathrm{r}^{-1/3}(\ln \mathrm{r})^{\kappa_\diamond}$ part of Y_Z where $1 - |\alpha|^2$ is not very small. To set the notation for the lemma, introduce v to denote the unit length vector field on the part of Y_Z where $|w| > 0$ that generates the kernel of w and has positive pairing with $*w$. A final bit of notation concerns the version of κ that appears in Lemma 7.2. The latter is denoted in what follows by $\kappa_\diamond$.

Lemma 7.8 *Assume that w is a harmonic 2-form with non-degenerate zeros. There exists $\kappa > \kappa_\diamond$ with the following significance: Fix $\mathrm{r} \geq \kappa$ and $\mu \in \Omega$ with $\mathcal{P}$-norm bounded by 1 and let (A, ψ) denote a solution to the (r, μ)-version of (2.4). Fix a positive integer k and set $m_k := (1 + \kappa^{-1})^k \kappa^2$. If $m_k < \mathrm{r}^{1/3}(\ln \mathrm{r})^{-\kappa}$, then there exists a set Θ_k, of at most κ segments of integral curves of v with the following properties:*

- *Each segment from Θ_1 is properly embedded in the $|w| \geq m_2^{-1}$ part of Y_Z and has length at most κ. Moreover, the union of the radius $\kappa \mathrm{r}^{-1/2}$ tubular neighborhoods of the segments in Θ_1 contain all points in the $|w| > \kappa^{-2}$ part of Y_Z where $1 - |\alpha|^2 > \frac{1}{4}\kappa_\diamond^{-1}$.*
- *If $k > 1$, then each segment from Θ_k is properly embedded in the $|w| \in [m_{k+1}^{-1}, m_{k-1}^{-1}]$ part of Y_Z and the union of the radius $\kappa m_k^{1/2} \mathrm{r}^{-1/2}$ tubular neighborhoods of the segments in Θ_k contain all $1 - |\alpha|^2 > \frac{1}{4}\kappa_\diamond^{-1}$ points in the $|w| \in [m_{k+1}^{-1}, m_{k-1}^{-1}]$ part of Y_Z.*

Proof. The proof has 8 steps. By way of a parenthetical remark, the proof follows a strategy like that used in Section IV.2c to prove Proposition IV.2.4.

Step 1: This step states a fact about the finite energy solutions to the vortex equations that plays a central role in the subsequent arguments. Keep in mind that a solution (A_0, α_0) is a finite energy solution when $1 - |\alpha_0|^2$ is an L^1-function. As noted in Part 1 of Section 7.2, if (A_0, α_0) is a finite energy solution then the integral of $1 - |\alpha_0|^2$ is 2π times a non-negative integer. Use m to denote this integer. As noted in Part 4 from Section 2b in [T2], there is a set $\vartheta \subset \mathbb{C}$ of m points with repetitions allowed such that

$$1 - |\alpha_0|^2 \leq c_0 \sum_{z \in \vartheta} e^{-\operatorname{dist}(\cdot, z)}. \tag{7.19}$$

This fact with Lemma 7.4 has a number of consequences with regards to the proof.

To say more, return to the context of Lemma 7.4. Let $\kappa_\diamond$ denote the version of the constant κ that appears in this lemma. Take $\mathrm{R} > \kappa_\diamond$ so as to apply the Lemma 7.4 when r is greater than the corresponding κ_{R}. With $\mathrm{r} \geq \kappa_{\mathrm{R}}$ and $\mu \in \Omega$ with $\mathcal{P}$-norm bounded by 1, let (A, ψ) denote a solution to the (r, μ)-version of (2.4). Fix $p \in Y_Z$ with $|w(p)| \geq \mathrm{r}^{-1/3}(\ln \mathrm{r})^{\kappa_\diamond}$ and use p to define the pair $(A_{\mathrm{r}}, \alpha_{\mathrm{r}})$ as instructed in Part 2 of Section 7.2. Assume for what follows that $1 - |\alpha_{\mathrm{r}}|^2 < \frac{1}{2}$ at distances between $\mathrm{R} + \kappa_\diamond(\ln \mathrm{R})^2$ and $\mathrm{R} - \kappa_\diamond(\ln \mathrm{R})^2$ from the origin in $\mathbb{C}$.

Lemma 7.4 asserts that $(A_{\mathrm{r}}, \alpha_{\mathrm{r}})$ has C^1-distance at most R^{-4} in the radius R disk about the origin in $\mathbb{C}$ from a finite energy vortex that defines a point in some $\mathrm{m} \leq \pi \mathrm{R}^2$ version of $\mathfrak{C}_{\mathrm{m}}$. Let (A_0, α_0) denote this solution. It follows from Lemma 7.4 that $1 - |\alpha_0|^2$ can be no greater than $\frac{1}{2} + 2\mathrm{R}^{-4}$ at the points in $\mathbb{C}$ with distance between R and $\mathrm{R} - \kappa_\diamond(\ln \mathrm{R})^2$ from the origin. This being the case, it follows from the bound $\mathrm{m} < \pi \mathrm{R}^2$ that the sum on the right hand side of (7.19) is no greater than R^{-4} at all points in $\mathbb{C}$ with distance between $\mathrm{R} - c_0 \ln \mathrm{R}$ and $\mathrm{R} - \kappa_\diamond(\ln \mathrm{R})^2 + c_0 \ln \mathrm{R}$ from the origin in $\mathbb{C}$. Granted this last conclusion, apply Lemma 7.4 to see that $1 - |\alpha_{\mathrm{r}}|^2$ is no greater than $2\mathrm{R}^{-4}$ at all points between $\mathrm{R} - c_0 \ln \mathrm{R}$ and $\mathrm{R} - \kappa_\diamond(\ln \mathrm{R})^2 + c_0 \ln \mathrm{R}$ from the origin in $\mathbb{C}$.

If $\mathrm{R} > c_0$, then the preceding conclusion implies that $1 - |\alpha|^2$ is bounded by $2\mathrm{R}^{-4}$ on the annulus in transverse disk centered at p with respective outer and inner radii given by $(\mathrm{R} - c_0 \ln \mathrm{R})\,(\mathrm{r}|w|(p))^{-1/2}$ and inner radius $(\mathrm{R} - \kappa_\diamond(\ln \mathrm{R})^2 + c_0 \ln \mathrm{R})\,(\mathrm{r}|w|(p))^{-1/2}$. Since α is nowhere vanishing on this annulus, the connection $\hat{A}_*$ is defined on this annulus by the same formula (3.7), and the last observation implies in particular that the connection $\hat{A}_*$ is flat and $\alpha|\alpha|^{-1}$ is $\hat{A}_*$-covariantly constant at points on this same annulus.

In the applications to come, the integer m will be bounded by c_0. If this is the case, then (7.19) with Lemma 7.4 implies that $\hat{A}_*$ is flat and $\alpha|\alpha|^{-1}$ is $\hat{A}_*$-covariantly constant at all point on the radius $(\mathrm{R} - c_0(\ln \mathrm{R})^2)(\mathrm{r}|w|(p))^{-1/2}$ transverse disk centered at p except at distance less than $c_0(\mathrm{r}|w|(p))^{-1/2}$ from a set of at most c_0 points.

Step 2: Fix $m_0 > c_0$ so that the $|w| \le m_0^{-1}$ part of Y_Z is a disjoint union of components with each component lying in the radius $c_0 m_0^{-1}$ ball about a zero of w. Require in addition that each such component lie in a Gaussian coordinate chart centered on the nearby zero of w as the embedded image of a closed ball in $\mathbb{R}^3$.

Fix $z > m_0$ and let κ_0 denote the sum of the versions of κ that appear in Lemmas 7.1, 7.2 and 7.7; and let κ_{z_0} denote the sum of κ_0 and the $\mathrm{R} = z^{10}$ version of the constant κ_{R} that appears in Lemma 7.4. But for cosmetic changes, the arguments in Section 6.4 of [T:W1] can be used with Lemmas 7.2, 7.4 and 7.6 plus what is said in Step 1 to find a z-independent $\kappa_1 \ge 100\kappa_0$ and a z-dependent $\kappa_z > \kappa_{z_0}$ such that the following is true:

Fix $\mathrm{r} \ge \kappa_z$ and $\mu \in \Omega$ with $\mathcal{P}$-norm bounded by 1. Suppose that (A, ψ) is a solution to the (r, μ)-version of (2.4). There exists a positive integer $n_0 < \kappa_1$ and a set Θ_0, of at most n_0 pairs of the form (γ, m) with γ being a properly embedded segment of an integral curve of v in the $|w| \ge z^{-6}$ part of Y_Z with length less than κ_1. Meanwhile, m is a positive integer. The set Θ_0 has the following additional properties:

(7.20)
- $\sum_{(\gamma,\mathrm{m})\in\Theta_0} \mathrm{m} \le \kappa_1$.
- Distinct curves from Θ_0 are separated by distance at least $\kappa_1 z^4 \mathrm{r}^{-1/2}$.
- If $p \in Y_Z$ is such that $|w(p)| \ge z^{-6}$ and $1 - |\alpha|^2 > \kappa_\diamond^{-1}$, then p has distance less than $z^4\mathrm{r}^{-1/2}$ from a curve in Θ_0.
- If $(\gamma, \mathrm{m}) \in \Theta_0$, then the integral of $\frac{i}{2\pi}F_{\hat{A}_*}$ over the radius $z^4\mathrm{r}^{-1/2}$ transverse disk centered at each point in γ is equal to m.

What follows is a parenthetical remark concerning the fourth bullet. The condition in the third bullet of (7.20) implies that $\alpha|\alpha|^{-1}$ is $\hat{A}_*$-covariantly constant near the boundary of the radius $z^4\mathrm{r}^{-1/2}$ transverse disk about each point in γ. It follows as a consequence that the integral of $\frac{i}{2\pi}F_{\hat{A}_*}$ over this disk is an integer; and it follows from Lemma 7.2 that this integer is non-negative. This being the case, the fourth bullet adds only that the integer is at least 1 and it is bounded a priori by a z, (A,ψ)-, μ- and r-independent number.

Step 3: Fix a ball $B \subset Y_Z$ centered on a zero of w that contains a component of the $|w| \le m_0^{-1}$ part of Y_Z. Suppose that $\varepsilon \in (0, 1)$ and that $z > m_0$ have been specified.

With κ_z as in Step 2, fix $\mathrm{r} \geq \kappa_z$, an element $\mu \in \Omega$ with $\mathcal{P}$-norm bounded by 1 and a solution, (A, ψ), to the (r, μ)-version of (2.4). Let k denote the largest integer with the properties listed below in (7.21). By way of notation, set $m_j := (1+\varepsilon)^j z^6$.

For each $j \in \{1, \ldots, k\}$, there exists $c_j \in (100, (100)^{2^{\kappa_1}})$ and a set, Θ_j, that consists of data sets which have the form $(\gamma, \mathrm{m}, \mathrm{D})$ with γ being a properly embedded segment of an integral curve of v in the $|w| \in [m_{j+1}^{-1}, m_{j-1}^{-1}]$ part of B, with m being a positive integer and with $\mathrm{D} \in (1, c_j)$. The set Θ_j has the following additional properties:

(7.21)
- $\sum_{\gamma, \mathrm{m}) \in \Theta_j} \mathrm{m} \leq \kappa_1$.
- Curves from distinct data sets in Θ_j are separated by distance at least $\frac{1}{2} c_j^2 z m_j^{1/2} \mathrm{r}^{-1/2}$.
- If $p \in Y_Z$ is such that $|w(p)| \in [m_{j+1}^{-1}, m_{j-1}^{-1})$ and $1 - |\alpha|^2 > \frac{1}{4}\kappa_0^{-1}$, then p has distance at most $\mathrm{D} z m_j^{1/2} \mathrm{r}^{-1/2}$ from a point on a curve from a datavset in Θ_j.
- If $(\gamma, m, \mathrm{D}) \in \Theta_j$, then the integral of $\frac{i}{2\pi} F_{\hat{A}_*}$ over the radius $\mathrm{D} z m_j^{1/2} \mathrm{r}^{-1/2}$ transverse disk centered at each point in γ is equal to m.

The next steps find (A, ψ), μ- and r-independent choices for ε and then z, and an (A, ψ), μ and r-independent $\kappa_* \geq \kappa_z$ such that $m_k \geq \mathrm{r}^{1/3} (\ln \mathrm{r})^{-\kappa_*}$ when r is greater than κ_*. Lemma 7.8 follows if such ε, z and κ_* exist.

The upcoming steps find the desired conditions on ε, z and the lower bound for r such that the conditions of the integer $k+1$ version of (7.21) are met if they are met for an integer k with $m_k < \mathrm{r}^{1/3} (\ln \mathrm{r})^{-2\kappa_0}$. This being the strategy, assume in what follows that k is such that $m_k < \mathrm{r}^{1/3} (\ln \mathrm{r})^{-2\kappa_0}$ and (7.21) holds.

Step 4: The A_*-directional covariant derivative along the vector field v is used momentarily to analyze the behavior of α at points along v's integral curves. This directional derivative is denoted in what follows by $(\nabla_A \alpha)_v$. The equations in (2.4) identify the latter with a linear combination of A_* -covariant derivatives of β. This being the case, Lemma 7.2 finds $|(\nabla_A \alpha)_v| \leq c_0 m [(1 - |\alpha|^2) + c_0 \mathrm{r}^{-1} m^3]^{1/2}$ on the $|w| > (2m)^{-1}$ part of Y_Z if $m \leq \mathrm{r}^{1/3} (\ln \mathrm{r})^{-\kappa_0}$. By way of a comparison, Lemma 7.2 bounds the norm of the remaining components of $\nabla_A \alpha$ by $c_0 m^{-1/2} \mathrm{r}^{-1/2} [(1 - |\alpha|^2) + c_0 \mathrm{r}^{-1} m^3]^{1/2}$.

What was said in the preceding paragraph about the norm of $|(\nabla_A \alpha)_v|$ has the following consequences for a point $p \in Y_Z$ where $|w| \in [m_{k+2}^{-1}, m_k^{-1}]$: Let γ_p denote the integral curve of v through p and let p' denote a point on the segment of γ_p where the distance

to p is less than $c_0^{-1}\kappa_\diamond^{-1}m_k^{-1}$.

$$
(7.22)\quad \begin{cases} \bullet \ \text{If } 1-|\alpha|^2 > \frac{1}{4}\kappa_\diamond^{-1} \text{ at } p, \text{ then } 1-|\alpha|^2 > \frac{1}{8}\kappa_\diamond^{-1} \text{ at } p'. \\ \bullet \ \text{If } 1-|\alpha|^2 \le \frac{1}{4}\kappa_\diamond^{-1} \text{ at } p, \text{ then } 1-|\alpha|^2 < \frac{1}{2}\kappa_\diamond^{-1} \text{ at } p'. \end{cases}
$$

This segment of γ_p is said in what follows to be the *short* segment of γ_p.

Note that if $\varepsilon \le c_0^{-1}\kappa_\diamond^{-2}$, then γ_p's short segment has points with $|w| > m_{k-1}^{-1}$. Assume in what follows that $\varepsilon \le c_0^{-1}\kappa_\diamond^{-2}$ is satisfied so as to invoke this fact about the short segment.

Step 5: This step constitutes a digression to supply a coordinate chart for any given $|w| > 0$ point in Y_Z that is used to exploit what is said in Step 4. To this end, suppose that $m > 1$ has been specified. Use I_m to denote the interval $[-c_0^{-1}m^{-1}, c_0^{-1}m^{-1}]$ and use D_m to denote the centered disk in $\mathbb{C}$ with radius $c_0^{-1}m^{-1}$. Use t to denote the coordinate for the interval I_m and use z for the complex coordinate on D_m. As will be explained momentarily, there is a coordinate chart embedding from $I_m \times D_m$ to Y_Z with the following properties:

$$
(7.23)\quad \begin{cases} \bullet \ \text{The point } (0,0) \text{ is mapped to } p \text{ and } I_m \times \{0\} \text{ is mapped to a segment of the integral curve of } v \text{ through } p. \\ \bullet \ \text{The image of any disk } \{t\} \times D_m \text{ is a transverse disk centered at the image of } (t,0). \\ \bullet \ \text{The function } z \mapsto |z| \text{ on } \{t\} \times D_m \text{ is the pull-back of the distance along the image of } \{t\} \times D_m \text{ to the image of } \{t,0\}. \\ \bullet \ \text{The vector field } v \text{ appears in these coordinates as } \frac{\partial}{\partial t} + \mathfrak{e} \text{ where } |\mathfrak{e}| \le c_0 m|z|. \end{cases}
$$

To construct such a coordinate chart, fix an isometric isomorphism between $K_*^{-1}|_p$ and $\mathbb{C}$. By way of a reminder, K_*^{-1} is used to denote the complex line bundle over the $|w(p)| > 0$ part of Y_Z whose underlying real bundle is the kernel of $*w$ with the complex structure defined using the metric and the restriction of the form w. Let γ_p again denote the integral curve of v through p. Parallel transport the resulting frame for K_*^{-1} along γ_p to identify K_*^{-1} along γ_p with $\gamma_p \times \mathbb{C}$. Fix a unit length affine parameter, t, for the segment of γ_p consisting of points with distance $c_0^{-1}m^{-1}$ or less from p with $t = 0$ corresponding to p. This identifies this segment with I_m. Granted this identification, compose the metric's exponential map from the I_m part of γ_p with the identification between K_*^{-1} on this segment and the product $\mathbb{C}$ bundle to define a

map from $I_m \times \mathbb{C}$ into Y_Z. The restriction of this map to $I_m \times D_m$ gives the desired coordinate embedding.

Step 6: Fix $p \in Y_Z$ such that $|w(p)| \in [m_{k+2}^{-1}, m_k^{-1}]$ and $1 - |\alpha|^2 > \frac{1}{4}\kappa_\diamond^{-1}$. Let p' denote a chosen point on Step 4's short segment of γ_p with $|w(p')| = m_{k+1}^{-1}$. It follows from (7.22) that $1 - |\alpha|^2 > \frac{1}{8}\kappa_\diamond^{-1}$ at p'. This being the case, it follows from Lemma 7.4 and Lemma IV.2.8 that if $z > c_0$ and if $\mathrm{r} > c_0$, then there is a point, with distance at most $c_0 m_{k+1}^{1/2}\mathrm{r}^{-1/2}$ from p' where $1 - |\alpha|^2 > \frac{1}{4}\kappa_\diamond^{-1}$. It then follows from the third bullet of (7.21) that there exists $(\gamma, \mathrm{m}, \mathrm{D}) \in \Theta_k$ such that p' has distance at most $(\mathrm{D}z + c_0)m_{k+1}^{1/2}\mathrm{r}^{-1/2}$ from a point in γ. Let p_* denote the latter point. Use the coordinate chart in (7.23) to see that short segment of γ_p intersects the transverse disk through p_* at a point with distance at most $(1 + c_0\varepsilon)(\mathrm{D}z + c_0)m_{k+1}^{1/2}\mathrm{r}^{-1/2}$ from p_*.

Extend the curves from Θ_k into the $|w| \geq m_{k+2}^{-1}$ part of Y_Z by integrating the vector field v. Use Γ_{k+1} to denote this set of extended curves. Given $\gamma \in \Gamma_{k+1}$, fix a point $p_\gamma \in \gamma$ where $|w| = m_{k+1}^{-1}$. The point p_γ has its corresponding version of the coordinate chart in (7.23) with γ appearing as an interval in the $z = 0$ locus that contains $(0, 0)$. Let I_γ denote this interval.

It follows from what was said in the preceding paragraph that the each point in B where $1 - |\alpha|^2 > \frac{1}{4}\kappa_\diamond^{-1}$ and $|w| \in [m_{k+2}^{-1}, m_k^{-1}]$ lies in the $|z| \leq (1 + c_0\varepsilon)(\mathrm{D}z + c_0)m_{k+1}^{1/2}\mathrm{r}^{-1/2}$ part of some $\gamma \in \Gamma_{k+1}$ version of $I_\gamma \times D_{m_{k+1}}$. In particular, if $\varepsilon < c_0^{-1}$ and $z > c_0$, then this subset is contained in the subset where $|z| < \frac{3}{2}\mathrm{D}z m_{k+1}^{1/2}\mathrm{r}^{-1/2}$. Assume that ε and z are such that this is the case.

Note in this regard that if $(\gamma, \mathrm{m}, \mathrm{D})$ and $(\gamma', \mathrm{m}', \mathrm{D}')$ are distinct elements in Θ_k, then the respective subsets of B that are parametrized via (7.23) by the $|z| \leq 2\mathrm{D}z m_{k+1}^{1/2}\mathrm{r}^{-1/2}$ part of $I_\gamma \times D_{m_{k+1}}$ and the $|z| \leq 2\mathrm{D}' z m_{k+1}^{1/2}\mathrm{r}^{-1/2}$ part of $I_{\gamma'} \times D_{m_{k+1}}$ are disjoint. This is a consequence of the second bullet in (7.21).

Step 7: Fix $(\gamma, \mathrm{m}, \mathrm{D}) \in \Theta_k$. It follows from what was said in Step 6 that $\alpha|\alpha|^{-1}$ is $\hat{A}_*$-covariantly constant in the solid annulus in $I_\gamma \times D_{m_{k+1}}$ that intersects any constant t slice as the annulus with inner radius $\frac{3}{2}\mathrm{D}z m_{k+1}^{1/2}\mathrm{r}^{-1/2}$ and outer radius $2\mathrm{D}z m_{k+1}^{1/2}\mathrm{r}^{-1/2}$. Granted this, it then follows from the third bullet of (7.21) that the integral of $\frac{i}{2\pi}F_{\hat{A}_*}$ over the $|z| < 2\mathrm{D}z m_{k+1}^{1/2}\mathrm{r}^{-1/2}$ part of any constant t disk in $I_\gamma \times D_{m_{k+1}}$ is the integer m.

To exploit the preceding observation, fix $t \in I_\gamma$ and let $p \in Y_Z$ denote the point that corresponds to $(t, 0) \in I_\gamma \times D_{m_{k+1}}$. Associate to p the pair $(A_\mathrm{r}, \alpha_\mathrm{r})$ as desribed in Part 2 of Section 7.2. Use c_z in what follows to denote a constant that is greater than

1 and depends only on z. It follows from Lemma 7.4 that if $z > c_0$ and if $\mathrm{r} > c_z$, then $(A_{\mathrm{r}}, \alpha_{\mathrm{r}})$ have C^1-distance less than z^{-10} on the radius $2\mathrm{D}z$ disk in $\mathbb{C}$ from a finite energy solution to the vortex equations. Moreover, what is said by Lemma 7.4 implies that such a finite energy solution must define a point in the space $\mathfrak{C}_{\mathrm{m}}$. Granted this, then (7.19) and Lemma 7.4 imply the following when $z > c_0$ and $\mathrm{r} > c_z$:

If $z > c_0$ and $\mathrm{r} \geq c_z$, then there is a set of at most n_0 points in the $|z| < \frac{3}{2}\mathrm{D}m_{k+1}^{1/2}\mathrm{r}^{-1/2}$ part of $\{t\} \times D_{m_{k+1}}$ such that

$$(7.24)\quad \begin{cases} \bullet\ \text{Each point is a zero of } \alpha. \\ \bullet\ \text{If } 1-|\alpha|^2 \geq \frac{1}{8}\kappa_\diamond^{-1} \text{ at } (t,z) \text{ and } |z| \leq 2\mathrm{D}\, m_{k+1}\mathrm{r}^{-1/2}\text{, then } z \text{ has distance} \\ \quad \text{at most } c_0 m_{k+1}^{1/2}\mathrm{r}^{-1/2} \text{ from some point in this set.} \end{cases}$$

Use $\vartheta_{\gamma,t}$ to denote this set of points and let $\mathfrak{U}_{\gamma,t}$ denote the set of connected components of the union of the disks of radius $c_0 m_{k+1}^{1/2}\mathrm{r}^{-1/2}$ about the points in $\vartheta_{\gamma,t}$. The next assertion is a $z > c_0$ and $\mathrm{r} > c_z$ consequence of (7.24) plus Lemma 7.4 and (7.19).

$$(7.25)\quad \begin{cases} \bullet\ \text{The connection } \hat{A}_* \text{ is flat and } \alpha|\alpha|^{-1} \text{ is } \hat{A}_*\text{-covariantly constant on the} \\ \quad \text{complement of } \bigcup_{U\in\mathfrak{U}_{\gamma,t}} U \text{ in the radius } 2\mathrm{D}m_{k+1}^{1/2}\mathrm{r}^{-1/2} \text{ disk about the} \\ \quad \text{origin in } \{t\} \times D_{m_{k+1}}. \\ \bullet\ \text{The integral of } \frac{i}{2\pi}F_{\hat{A}_*} \text{ over any set } U \in \mathfrak{U}_{\gamma.t} \text{ is a positive integer; and} \\ \quad \text{the sum of these integers is equal to m.} \end{cases}$$

The next step constructs Θ_{k+1} with the help of the various $(\gamma, \mathrm{m}, \mathrm{D}) \in \Theta_k$ versions of $\vartheta_{\gamma,0}$.

Step 8: To construct Θ_{k+1}, it is necessary to cluster the points from the various $(\gamma, \mathrm{m}, \mathrm{D}) \in \Theta_k$ versions of $\vartheta_{\gamma,0}$ so that points in the same cluster are pairwise much closer to each other than they are to any point in another cluster. This is necessary so as to find the desired constant c_{k+1} for the integer $k+1$ version of (7.21). An appropriate clustering can be found by invoking Lemma 2.12 in [T4]. In particular, an appeal to this lemma finds $c_{k+1} \in (100, (100)^{2^{\kappa_1}})$ and a set of at most κ_1 pairs of the form (p, D) where $p \in B$ is such that $|w(p)| = m_{k+1}^{-1}$ and where $\mathrm{D} \in (1, c_{k+1})$. This set is denoted by ϑ and it has the properties in the list that follows.

$$(7.26)\quad \begin{cases} \bullet\ \text{If } (p,\mathrm{D}) \text{ and } (p',\mathrm{D}') \text{ are distinct elements in } \vartheta\text{, then } \operatorname{dist}(p,p') > \\ \quad c_{k+1}^2 z\, m_{k+1}^{1/2}\mathrm{r}^{-1/2}. \\ \bullet\ \text{If } p \text{ corresponds via (7.23) to a point in some } (\gamma,\mathrm{m},\mathrm{D}) \in \Theta_k \text{ version of} \\ \quad \vartheta_{k,0}\text{, then } p \text{ has distance at most } \frac{1}{4}\mathrm{D}z\, m_{k+1}^{1/2}\mathrm{r}^{-1/2} \text{ from a point of some} \\ \quad \text{pair from } \vartheta. \end{cases}$$

Note for future reference that the bound in the first bullet of (7.26) has the following implication when $z > c_0$ and $\mathrm{r} > c_z$:

(7.27) If (p, D) and (p', D') are distinct elements in ϑ, then the distance between any two points on the respective short segments γ_p and $\gamma_{p'}$ is greater than $\frac{1}{2} c_{k+1}^2 z m_{k+1}^{1/2} \mathrm{r}^{-1/2}$.

It follows from (7.25) and (7.26) that if $(\gamma, \mathrm{m}, \mathrm{D}) \in \Theta_k$ and if $U \in \mathfrak{U}_{\gamma,0}$, then U is in the transverse disk of radius $\frac{1}{2} \mathrm{D} z m_{k+1}^{1/2} \mathrm{r}^{-1/2}$ centered at a point of some pair in ϑ. Granted this last conclusion, then the next assertion is a direct consequence of what is said in Step 4 if $z > c_0$ and $\mathrm{r} > c_z$.

(7.28) If $(\gamma, m, \mathrm{D}) \in \Theta_k$ and $t \in I_\gamma$, then each $U \in \mathfrak{U}_{\gamma,t}$ is contained in the radius $\mathrm{D}\, z\, m_{k+1}^{1/2} \mathrm{r}^{-1/2}$ tubular neighborhood of the integral curve of v through a point of some pair from ϑ.

Let $(p, \mathrm{D}) \in \vartheta$. What is said in (7.27) and (7.28) has the following consequence:

(7.29) The integral of $\dfrac{i}{2\pi} F_{\hat{A}_*}$ on the radius $\mathrm{D}\, z\, m_{k+1}^{1/2} \mathrm{r}^{-1/2}$ transverse disk about any point in the $|w| \in [m_{k+2}^{-1}, m_k^{-1}]$ part of γ_p is a positive integer.

Let m denote now this integer.

Define Θ_{k+1} to be the set $\{(p, m, \mathrm{D}) \,|\, (p, \mathrm{D}) \in \vartheta)\}$. It follows from (7.25) and (7.27)-(7.29) that the requirements for the integer $k+1$ version of (7.21) are met using c_{k+1} and the set Θ_{k+1} if $\varepsilon < c_0$, $z > c_0$ and $\mathrm{r} > c_z$. □

7.6 The spectral flow function

This subsection constitutes a digression to say more about the definition of $\mathfrak{f}_s$. Each pair $\mathfrak{c} = (A, \psi)$ in $\mathrm{Conn}(E) \times C^\infty(Y_Z; \mathbb{S})$ and a given real number z determine an associated, unbounded, self-adjoint operator on $L^2(Y_Z; iT^*Y_Z \oplus \mathbb{S} \oplus i\mathbb{R})$. This operator is denoted by $\mathfrak{L}_{\mathfrak{c},z}$ and it is defined as follows: A given smooth section $\mathfrak{h} = (b, \eta, \phi)$ of $iT^*Y_Z \oplus \mathbb{S} \oplus i\mathbb{R}$ is sent by $\mathfrak{L}_{\mathfrak{c},z}$ to the section whose respective iT^*Y_Z, $\mathbb{S}$, and $i\mathbb{R}$-summands are

$$(7.30)\qquad \begin{cases} *db - d\phi - 2^{-1/2} z^{1/2} (\psi^\dagger \tau \eta + \eta^\dagger \tau \psi), \\ D_A \eta + 2^{1/2} z^{1/2} (\mathrm{cl}(b)\psi + \phi\psi), \\ *d * b - 2^{-1/2} z^{1/2} (\eta^\dagger \psi - \psi^\dagger \eta). \end{cases}$$

The spectrum of this operator is discrete with no accumulation points and has finite multiplicity. The spectrum is also unbounded from above and unbounded from below.

The section ψ_E of $\mathbb{S}$ is chosen so that the (A_E, ψ_E) and $z = 1$ version of (7.30) has trivial kernel. If the $z = \mathrm{r}$ and $\mathfrak{c} = (A, \psi)$ version of (7.30) has trivial kernel, then the value of the spectral flow function $\mathfrak{f}_s(\mathfrak{c})$ is a certain algebraic count of the number of zero eigenvalues that appear along a continuous path $\mathfrak{d}$ of operators that start at the $z = 1$ and (A_E, ψ_E) version of (7.30) and end at the $z = \mathrm{r}$ and (A, ψ) version and such that each member of the path differs from $\mathfrak{L}_{\mathfrak{c},\mathrm{r}}$ by a bounded operator on $L^2(Y_Z; iT^*Y_Z \oplus \mathbb{S} \oplus i\mathbb{R})$. For the purposes of the definition, it is sufficient to consider paths that are parametrized by $[0, 1]$ such that the following conditions are met: Let $\vartheta \subset [0, 1]$ denote the parameters that label an operator with zero as an eigenvalue. Then ϑ is finite and in each case, the zero eigenvalue has multiplicity 1 and the zero eigenvalue crossing is transversal as the parameter varies in a small neighborhood of the given point in $[0, 1]$. Having chosen such a path, a given point in the corresponding version of ϑ contributes either $+1$ or -1 to $\mathfrak{f}_s(\mathfrak{c})$. The contribution is $+1$ when the eigenvalue crosses zero from negative value to positive value as the parameter in $[0, 1]$ varies near the given point in ϑ; and it contributes -1 to $\mathfrak{f}_s(\mathfrak{c})$ if the eigenvalue crosses zero from a positive value to negative value near the given point.

If $\mathfrak{L}_{\mathfrak{c},\mathrm{r}}$ has non-trivial kernel, then $\mathfrak{f}_s(\mathfrak{c})$ is defined in the upcoming (7.31). The definition uses the following terminology: Given $\varepsilon > 0$, and $\mathfrak{c} \in \mathrm{Conn}(E) \times C^\infty(Y_Z; \mathbb{C})$, the definition uses $\mathfrak{N}_\varepsilon(\mathfrak{c})$ to denote the subset of pairs in $\mathrm{Conn}(E) \times C^\infty(Y_Z; \mathbb{C})$ with the following two properties: A pair $\mathfrak{c}'$ is in $\mathfrak{N}_\varepsilon(\mathfrak{c})$ if it has C^1-distance less than ε from $\mathfrak{c}$, and if $\mathfrak{L}_{\mathfrak{c}',\mathrm{r}}$ has trivial kernel. Standard perturbation theory for ellipitic operators proves that $\mathfrak{N}_\varepsilon(\mathfrak{c})$ is non-empty for any $\varepsilon > 0$. With the notation set, define $\mathfrak{f}_s(\mathfrak{c})$ by the rule

$$\mathfrak{f}_s(\mathfrak{c}) = \limsup_{\varepsilon \to 0} \{\, \mathfrak{f}_s(\mathfrak{c}') \,|\, \mathfrak{c}' \in \mathfrak{N}_\varepsilon(\mathfrak{c})\}. \tag{7.31}$$

Note by the way that the lim sup in (7.31) differs from the corresponding lim inf by the dimension of the kernel of $\mathfrak{L}_{\mathfrak{c},\mathrm{r}}$.

7.7 The L^1-norm of B_A, the spectral flow and the functions $\mathfrak{cs}^{\mathfrak{f}}$, $\mathrm{w}^{\mathfrak{f}}$, $\mathfrak{a}^{\mathfrak{f}}$

The functions

$$\mathfrak{cs}^{\mathfrak{f}} = \mathfrak{cs} - 4\pi^2 \mathfrak{f}_s, \quad \mathrm{w}^{f} = \mathrm{w} - 2\pi \mathfrak{f}_s \quad \text{and} \quad \mathfrak{a}^{\mathfrak{f}} = \mathfrak{a} + 2\pi(\mathrm{r} - \pi)\mathfrak{f}_s \tag{7.32}$$

are invariant under $C^\infty(Y_Z; S^1)$ action on $\mathrm{Conn}(E) \times C^\infty(Y_Z; \mathbb{S})$ that has $\hat{u} \in C^\infty(Y_Z; S^1)$ sending (A, ψ) to $(A - \hat{u}^{-1} d\hat{u}, \hat{u}\psi)$. The upcoming Lemma 7.9 supplies a priori bounds

on the values of these functions when evaluated on solutions to a given (r,μ)-version of (2.4). It also gives a better bound for the L^1-norm of the curvature of the connection component of a solution than the bound in Lemma 7.6.

Lemma 7.9 *Suppose that w is a harmonic 2-form with non-degenerate zeros. There exists $\kappa > \pi$ with the following significance: Fix $\mathrm{r} \geq \kappa$ and a 1-form $\mu \in \Omega$ with $\mathcal{P}$-norm less than 1. Suppose that (A,ψ) is a solution to the (r,μ)-version of (2.4). Then:*

- *The L^1-norm of B_A is no greater than $\kappa(\ln \mathrm{r})^4$.*
- $|\mathfrak{cs}^{\mathfrak{f}}| < \mathrm{r}^{6/7}$,
- $|\mathrm{w}^{\mathfrak{f}}| < \mathrm{r}^{6/7}$,
- $|\mathfrak{a}^{\mathfrak{f}}| < \mathrm{r}^{13/14}$.

As a parenthetical remark, the precise powers of r that appear in the last three bullets are significant with regards to the applications to come only to the extent that the power is less than 1 in the second and third bullets and so less than 2 in the final bullet.

Proof. By way of a look ahead, what is said in Lemma 7.8 plays a vital role in the proof of all four bullets. The proof of Lemma 7.9 has 10 parts.

Part 1: The proof of Lemma 7.9's first bullet has four steps. To set the notation for the proof, introduce κ_* to denote the version of the constant κ that appears in Lemma 7.8. As in Lemma 7.8, set $m_k = (1+\kappa_*^{-1})\kappa_*^2$ for $k \in \{1,2,\ldots\}$. Assume in what follows that k is such that $m_k < \mathrm{r}^{1/3}(\ln \mathrm{r})^{-\kappa_*}$.

Step 1: Use the first bullet of Lemma 7.8 and the fourth bullet of Lemma 7.2 to see the $|B_A| < c_0$ at points in the $|w| > m_1^{-1}$ part of Y_Z where the distance to all segments in Θ_1 is greater than $c_0(\ln \mathrm{r})^2\mathrm{r}^{-1/2}$. This understood, this part of Y_Z contributes at most c_0 to the L^1-norm of B_A. Meanwhile, the $|w| > m_1^{-1}$ part of Y_Z of the union of the radius $c_0(\ln \mathrm{r})^2\mathrm{r}^{-1/2}$ tubular neighborhoods of the segments in Y_Z contributes at most $c_0(\ln \mathrm{r})^4$ to the L^1-norm of B_A.

Step 2: Fix $k > 1$. Use the integer k version of the second bullet of Lemma 7.8 with the fourth bullet of Lemma 7.2 to see that $|B_A|$ is bounded by $c_0(1+m_k^2)$ at points in the $|w| \in [m_k^{-1}, m_{k-1}^{-1}]$ part of Y_{Z-} where the distance to all segments in Θ_k is greater than $c_0 m_k^{1/2}(\ln \mathrm{r})^2\mathrm{r}^{-1/2}$. Since this subset of Y_Z has volume at most $c_0 m_k^{-3}$, so this portion of the $|w| \in [m_k^{-1}, m_{k-1}^{-1}]$ subset in Y_Z contributes at most $c_0 m_k^{-1}$ to the L^1-norm of B_A. The volume of the remaining part of the $|w| \in [m_k^{-1}, m_{k-1}^{-1}]$ subset in Y_{Z-} is at most $c_0\mathrm{r}^{-1}(\ln \mathrm{r})^4$. Indeed, this can be seen from (7.23) using the fact that each segment

in Θ_k has length at most $c_0 m_k^{-1}$. As $|B_A|$ is no greater than $c_0 m_k^{-1}\mathrm{r}$ on this part of Y_Z, so this part of Y_Z contributes at most $c_0 m_k^{-1}(\ln \mathrm{r})^4$ to the L^1-norm of B_A.

Step 3: Lemma 7.3 implies that $|B_A|$ is bounded by $c_0\mathrm{r}^{2/3}(\ln \mathrm{r})^{\kappa_*}$ on the subset of Y_Z where $|w| \leq c_0\mathrm{r}^{-1/3}(\ln \mathrm{r})^{\kappa_*}$. The volume of this subset is at most $\mathrm{r}^{-1}(\ln \mathrm{r})^{3\kappa_*}$ and so the contribution from this part of Y_Z to the L^1-norm of B_A is no greater than $c_0\mathrm{r}^{-1/4}$.

Step 4: Sum the bounds in Steps 1-3 to see that the L^1-norm of B_A is no greater than $c_0(\ln \mathrm{r})^4 \sum_{k=0,1,\ldots}(1+1/\kappa_*)^{-k}$. This sum is bounded by $c_0\kappa_*(\ln \mathrm{r})^4$.

Part 2: The proof of the last three bullets of the lemma starts with the following observation: There is a smooth map, $\hat{u}\colon Y_Z \to S^1$, such that the connection $A' = A - \hat{u}^{-1}d\hat{u}$ can be written as $A' = A_E + \hat{a}_{A'}$ where $\hat{a}_{A'}$ is a coclosed, $i\mathbb{R}$-valued 1-form whose L^2 orthogonal projection to the space of harmonic 1-forms on Y_Z is bounded by c_0. The upcoming Lemma 7.10 asserts the pointwise bound $|\,\hat{a}_{A'}\,| \leq c_0\mathrm{r}^{1/3}(\ln \mathrm{r})^{c_0}$. Assume this bound for the time being.

Introduce $\mathfrak{c}'$ to denote $(A - \hat{u}^{-1}d\hat{u}, \hat{u}\psi)$. The supremum bound for $|\,\hat{a}_{A'}\,|$ and the L^1-bound for B_A from Lemma 7.9's first bullet imply directly that $|\mathfrak{cs}(\mathfrak{c}')| \leq c_0\mathrm{r}^{1/2}(\ln \mathrm{r})^{c_0}$. The L^1-bound for B_A also implies that $|\mathrm{W}(\mathfrak{c}')| \leq c_0(\ln \mathrm{r})^{c_0}$. Thus, $|\mathfrak{a}(\mathfrak{c}')| \leq c_0\mathrm{r}(\ln \mathrm{r})^{c_0}$. Granted these bounds, then the last three bullets of Lemma 7.9 follow if

$$|\mathfrak{f}_s(\mathfrak{c}')| \leq \mathrm{r}^{6/7}. \tag{7.33}$$

The fact that (7.33) holds given the assumptions of the lemma is proved in the remaining parts of this subsection.

Part 3: The proof of the last three bullets of Lemma 7.9 invoked a pointwise bound for $|\,\hat{a}_{A'}\,|$. The lemma that follows supplies the asserted bound.

Lemma 7.10 *There exists $\kappa > \pi$ with the following significance: Fix $\mathrm{r} \geq \kappa$ and an element $\mu \in \Omega$ with $\mathcal{P}$-norm less than 1. Let (A, ψ) denote a solution to the (r, μ)-version of (2.4). Write A as $A_E + \hat{a}_A$ and assume that $\hat{a}_A$ is a coclosed 1-form. Use c to denote the L^2-norm of the L^2-orthogonal projection of $\hat{a}_A$ to the space harmonic 1-forms. Then $|\,\hat{a}_A\,| \leq \mathrm{r}^{1/2}(\ln \mathrm{r})^{\kappa} + \kappa c$.*

Proof. The proof that follows has three steps.

Step 1: Write $\hat{\mathrm{a}}_A$ as $\hat{\mathrm{a}}^{\perp} + \mathfrak{p}$ where $\hat{\mathrm{a}}^{\perp}$ is L^2-orthogonal to the space of harmonic 1-forms and where $\mathfrak{p}$ is a harmonic 1-form. The norm of $\mathfrak{p}$ is bounded by $c_0 c$. To bound $\hat{\mathrm{a}}^{\perp}$, let $\mathcal{C}^{\perp} \subset C^{\infty}(Y_Z; T^*Y_Z)$ denote the subspace of coclosed 1-forms that are L^2-orthogonal to the space of harmonic 1-forms. The operator $*d$ maps $\mathcal{C}^{\perp}$ to itself and Hodge theory gives a Green's function inverse. Given $p \in M$, the corresponding Green's function with pole at p is denoted by $G_p^{\perp}(\cdot)$. This function is smooth on the complement of p and it obeys the pointwise bound $|G_p^{\perp}(\cdot)| \leq c_0 \,\mathrm{dist}\,(\cdot, p)^{-2}$.

Step 2: Introduce κ_* to denote Lemma 7.8's version of κ. Reintroduce from Lemma 7.8 the sequence $\{m_k = (1+\kappa_*^{-1})^k \kappa^2\}_{k=1,2\ldots,N}$ with N being the greatest integer such that $m_k < \mathrm{r}^{1/3}(\ln \mathrm{r})^{-\kappa_*}$. Let $\mathcal{U}_1$ denote the $|w| > m_2^{-1}$ part of Y_Z. For $k \in \{1, \ldots, N-1\}$, use $\mathcal{U}_k$ to denote the $|w| \in [m_{k+1}^{-1}, m_{k-1}^{-1}]$ part of Y_Z, and use $\mathcal{U}_N$ to denote the part of Y_Z where $|w| \leq m_{N-1}^{-1}$. Given $k \in \{1, \ldots, N-1\}$, let Γ_k denote the set of curves from Θ_k's data sets. By way of a reminder, there are at most κ_* curves in Γ_k and each is a properly embedded segment of an integral curve of v in $\mathcal{U}_k$.

Lemmas 7.2 and 7.8 supply $c_* \in (1, c_0)$ with the following property: If $p \in \mathcal{U}_k$ has distance greater than $c_* m_k \mathrm{r}^{-1/2}(\ln \mathrm{r})^2$ to any curve from Γ_k, then $1 - |\alpha|^2 \leq c_0 m_k^3 \mathrm{r}^{-1}$. Denote by $\mathcal{T}_{k1}$ the union of the radius $c_* m_k \mathrm{r}^{-1/2}(\ln \mathrm{r})^2$ tubular neighborhoods of the curves from Γ_k. Since $*d\,\hat{\mathrm{a}}^{\perp} = B_A$, it follows from Lemmas 7.1 and 7.2 that $|B_A| \leq c_0 m_k^2$ on $\mathcal{U}_k - \mathcal{T}_{k1}$, and it follows from Lemma 7.2 and Lemma 7.3 that $|B_A| \leq c_0 m_k^{-1} \mathrm{r}$ on $\mathcal{T}_{k1}$. Note also that the volume of $\mathcal{U}_k$ is at most $c_0 m_k^{-3}$ and that of $\mathcal{T}_{k1}$ at most $c_0 m_k^{-1} \mathrm{r}^{-1}(\ln \mathrm{r})^4$.

Step 3: Suppose that $k \in \{1, \ldots, N-1\}$ and that $p \in \mathcal{U}_k$. Keeping in mind that the volume of $\mathcal{U}_k$ is bounded by $c_0 m_k^{-3}$, it follows from what is said about $G_p^{\perp}$ in Step 1 and what is said about $|B_A|$ in Step 2 that

$$|\,\hat{\mathrm{a}}^{\perp}\,|(p) \leq c_0 \int_{\mathcal{T}_k} \mathrm{dist}\,(\cdot, p)^{-2} |B_A| + c_0 (m_k + (\ln \mathrm{r})^{c_0}). \tag{7.34}$$

Use the various $\gamma \in \Gamma_k$ versions of (7.23) to see that the integral on the right hand side of (7.34) is no greater than $c_0 m_k^{-1/2} \mathrm{r}^{1/2} (\ln \mathrm{r})^{c_0}$.

Suppose that $p \in \mathcal{U}_N$. In this case, what is said about $G_p^{\perp}$ in Step 1 and what is said in Step 2 about $|B_A|$ imply that $|\,\hat{\mathrm{a}}^{\perp}\,|(p) \leq c_0 \mathrm{r}^{1/3} (\ln \mathrm{r})^{c_0}$. □

Part 4: Fix $c > 1$ and suppose that $\mathfrak{c} = (A, \psi)$ solves (2.4) and is such that the $i\mathbb{R}$-valued 1-form $\hat{a}_A = A - A_E$ is coclosed and that the L^2-norm of its L^2-orthogonal projection to the space of harmonic 1-forms on Y_Z is less than c. The value of $\mathfrak{f}_s$ will be computed by choosing a convenient, piecewise continuous path of self-adjoint operator from the (A_E, ψ_E) and $z = 1$ version of (7.30) to $\mathfrak{L}_{c,\mathrm{r}}$. This path is the concatentation of the three real analytic segments that are described below. The absolute value of $\mathfrak{f}_s(\mathfrak{c})$ is no greater than the absolute value of the sum of the absolute values of the spectral flow along the three segments.

By way of notation, each segment is parametrized by $[0, 1]$ and the operator labeled by a given $s \in [0, 1]$ in the k'th segment is denoted by $\mathcal{L}_{k,s}$. The first segment's operator $\mathcal{L}_{1,s}$ for $s \in [0, 1]$ is the (A_E, ψ_E) and $z = 1 - s$ version of (7.30). This path has no dependence on (A, ψ) or r, and so the absolute value of the spectral flow along this path is no greater than c_0. The remaining two segments are:

(7.35)
- The second segment's operator $\mathcal{L}_{2,s}$ for $s \in [0, 1]$ is the $(A_E + s\ \hat{a}_A, 0)$ version of (7.30).
- The third segment's operator $\mathcal{L}_{3,s}$ for $s \in [0, 1]$ is the (A, ψ), $z = s^2\mathrm{r}$-version of (7.30).

The strategy for bounding the absolute value of the spectral flow along (7.35)'s two segments borrows heavily from Section 3 of [T:W2]. To say more about this, suppose that $\mathcal{L}$ is an unbounded, self-adjoint operator on a given separable Hilbert space with discrete spectrum with no accumulation points and finite multiplicities. Let $\{e_s\}_{s\in[0,1]}$ denote a real analytic family of bounded, self-adjoint operators on this same Hilbert space. Of interest is the spectral flow between the $s = 0$ and $s = 1$ members of the family $\{\mathcal{L}_s = \mathcal{L} + e_s\}_{s\in[0,1]}$. To obtain a bound, fix for the moment $T > 0$ and let $\mathfrak{n}_{T,s}$ denote the number of linearly independent eigenvectors of $\mathcal{L}_s$ whose eigenvalue has absolute value no greater than T. Set $\mathfrak{n}_T = \sup\{\mathfrak{n}_{T,s}\}_{s\in[0,1]}$. As explained in [T:W3], the spectral flow for the family $\{\mathcal{L}_s\}_{s\in[0,1]}$ has absolute value no greater than

$$\frac{1}{2T}\mathfrak{n}_T \sup\left\{\left\|\frac{d}{ds}e_s\right\|_{op}\right\}_{s\in[0,1]}, \tag{7.36}$$

where the norm $\|\cdot\|_{op}$ here denotes the operator norm.

The supremum in (7.36) for the family $\{\mathcal{L}_{2,s}\}_{s\in[0,1]}$ is bounded by $c_0|\,\hat{a}_{A'}\,|$, and thus by $c_0\mathrm{r}^{1/2}(\ln \mathrm{r})^{c_0}$. It follows from Lemma 7.3 that the supremum that appears in (7.36) for the family $\{\mathcal{L}_{3,s}\}_{s\in[0,1]}$ is $c_0\mathrm{r}^{1/2}$. This understood, then (7.36) in either case leads

to

(7.37) The absolute value of the spectral flow along the families $\{\mathcal{L}_{2,s}\}_{s\in[0,1]}$ and $\{\mathcal{L}_{3,s}\}_{s\in[0,1]}$ is no greater than $c_0 \mathrm{r}^{1/2}(\ln \mathrm{r})^{c_0}\frac{1}{T}\mathfrak{n}_T$.

The next part of the subsection describes the strategy that is used to bound $\mathfrak{n}_T$ for a suitable choice of T.

Part 5: A bound for $\mathfrak{n}_T$ is obtained with the help of the Weitzenböck formula in (IV.A.12) for a given $z \geq 0$ version of $\mathfrak{L}^2_{\mathfrak{c},z}$. This formula writes $\mathfrak{L}^2_{\mathfrak{c},z}$ as $\nabla_A^\dagger \nabla_A + \mathcal{Q}$ where $\mathcal{Q}$ denotes an endomorphism of $iT^*Y_Z \oplus \mathbb{S} \oplus i\mathbb{R}$ and ∇_A denotes here the connection on the bundle $iT^*Y_Z \oplus \mathbb{S} \oplus i\mathbb{R}$ given by the Levi-Civita connection on the iT^*Y_Z-summand, the Levi-Civita connection and A on the $\mathbb{S}$ summand, and the product connection on the $i\mathbb{R}$ summand. This rewriting of $\mathfrak{L}^2_{\mathfrak{c},z}$ is used to write the square of the L^2-norm of $\mathfrak{L}_{\mathfrak{c},z}\mathfrak{q}$ as

$$\int_{Y_Z} |\mathfrak{L}_{\mathfrak{c},z}\mathfrak{q}|^2 = \int_{Y_Z} \left(|\nabla_A \mathfrak{q}|^2 + \langle \mathfrak{q}, \mathcal{Q}\mathfrak{q}\rangle\right), \tag{7.38}$$

with $\langle \cdot,\cdot\rangle$ denoting here the Hermitian inner product on $iT * Y_Z \oplus \mathbb{S} \oplus i\mathbb{R}$. If $\mathfrak{q}$ is a linear combination of eigenvectors of $\mathfrak{L}_{\mathfrak{c},z}$ with the norm of the eigenvalue bounded by T, then what is written in (7.38) is no greater than T^2 times the square of the L^2-norm of $\mathfrak{q}$.

The formula in (7.38) is exploited to bound $\mathfrak{n}_T$ using the following observation: Suppose that $\mathfrak{U}$ is an open cover of Y_Z such that no point is contained in more than c_0 sets from $\mathfrak{U}$. Let h denote for the moment a given function on Y_Z. Then

$$\int_{Y_Z} h^2 \leq \sum_{U\in\mathfrak{U}} \int_U h^2 \leq c_0 \int_{Y_Z} h^2. \tag{7.39}$$

Hold onto this last observation for the moment. Use $c_\diamond$ to denote the version of c_0 that appears in this last inequality.

The endomorphism $\mathcal{Q}$ is self-adjoint, so it can be written at any given point as a sum $\mathcal{Q}^+ + \mathcal{Q}^-$ with $\mathcal{Q}^+$ being positive semi-definite and $\mathcal{Q}^-$ being negative definite. With this fact in mind, suppose now that each set $U \in \mathfrak{U}$ has an assigned, finite dimensional vector subspace $V_U \in C^\infty(U; iT^*M \oplus \mathbb{S} \oplus i\mathbb{R})$ with the following property:

(7.40) If $q \in C^\infty(U; iT^*M \oplus \mathbb{S} \oplus i\mathbb{R})$ is L^2-orthogonal to V_U, then
$$\int_U \left(|\nabla_A \mathfrak{q}|^2 + \langle \mathfrak{q}, \mathcal{Q}^+\mathfrak{q}\rangle\right) > 2c_\diamond(T^2 + \sup_U |\mathcal{Q}^-|) \int_U |\mathfrak{q}|^2.$$

Given V_U, define $\Phi_U\colon C^\infty(Y_Z; iT^*M \oplus \mathbb{S} \oplus i\mathbb{R}) \to V_U$ to be the composition of first restriction to U and then the L^2-orthogonal projection. Set $\mathcal{V} = \bigoplus_{U\in\mathfrak{U}} V_U$ and denote by Φ the linear map from $C^\infty(Y_Z; iT^*M \oplus \mathbb{S} \oplus i\mathbb{R})$ to $\mathcal{V}$ given by $\bigoplus_{U\in\mathfrak{U}} \Phi_U$.

The inequalities in (7.39) and (7.40) have the following immediate consequence: If $\mathfrak{q} \in \operatorname{Ker}(\Phi)$, then the L^2-norm of $\mathfrak{L}_{\mathfrak{c},z}$ is greater than T. Given that such is the case, it then follows directly that $\mathfrak{n}_T \le \sum_{U\in\mathfrak{U}} \dim(V_U)$.

The subsequent parts of the proof define a version of $\mathfrak{U}$ for suitable T with associated vector spaces $\{V_U\}_{U\in\mathfrak{U}}$ such that (7.40) holds. The resulting bound for $\mathfrak{n}_T$ leads via (7.37) to the bound in (7.33) for $|\mathfrak{f}_s|$.

Part 6: Part 5 alludes to a certain open cover of Y_Z. This part of the subsection defines this cover. To this end, reintroduce from Step 2 of the proof of Lemma 7.10 the sets $\{\mathcal{U}_k\}_{1\le k\le N}$. The cover in question is given as $\mathfrak{U} = \bigcup_{k=1,2,\ldots N} \mathfrak{U}_k$ where all $U \in \mathfrak{U}_k$ are subsets of $\mathcal{U}_{k-1} \cup \mathcal{U}_k \cup \mathcal{U}_{k+1}$. The definition requires the choice of a constant $c > 1$. Part 10 of the proof gives a lower bound for c by c_0. Any choice above this bound suffices.

To define a given $k \in \{1, \ldots, N-1\}$ version of $\mathfrak{U}_k$, reintroduce from Step 2 of the proof of Lemma 7.10 the set Γ_k, this being the set of curves from Θ_k's data sets. By way of a reminder, there are at most κ_* curves in Γ_k and each is a properly embedded segment of an integral curve of v in $\mathcal{U}_k$. This same step in the proof of Lemma 7.8 introduced a constant c_* such that $1 - |\alpha|^2 < c_0 m_k^3 \mathrm{r}^{-1}$ at points with distance $c_* m_k^{1/2} \mathrm{r}^{-1/2} (\ln \mathrm{r})^2$ or more to all curves from Γ_k. The discussion that follows uses R_k to denote $c_* m_k^{1/2} \mathrm{r}^{-1/2} (\ln \mathrm{r})^2$ and ρ_k to denote $c^{-1} \min(T, m_k^{-1})$.

The collection $\mathfrak{U}_k$ for $k \in \{1, \ldots, N-1\}$ is written as $\mathfrak{U}_{k-} \cup \mathfrak{U}_{k0} \cup \mathfrak{U}_{k+}$. The sets from $\mathfrak{U}_{k-}$ are balls of radius ρ_k whose centers have distance at least ρ_k to all curves from Γ_k. These balls cover the complement in $\mathcal{U}_k$ of the union of the radius $2\rho_k$ tubular neighborhoods of the curves from Γ_k. A cover as just described can be found with less than $c_0 \rho_k^{-3} m_k^{-3}$ balls, and such is the case with the cover $\mathfrak{U}_{k-}$.

The sets from $\mathfrak{U}_{k0}$ are balls with distance between $2\rho_k$ and R_k to at least one curve from Γ_k. Let U denote a give ball from $\mathfrak{U}_{k0}$ and let D denote its distance to the union of the curves from Γ_k. The radius of U is equal to $\frac{1}{8}\mathrm{D}$. The various $\gamma \in \Gamma_k$ versions of (7.23) can be used to see that a collection of $c_0 \ln(\rho_k/\mathrm{R}_k)(\mathrm{R}_k m_k)^{-1}$ balls of this sort can be found whose union contains every point in $\mathcal{U}_k$ with distance between ρ_k and $2\mathrm{R}_k$ to at least one curve from Γ_k. The set $\mathfrak{U}_{k0}$ is such a collection of balls.

The set $\mathfrak{U}_{k+}$ consists of balls of radius $c^{-1}m_k^{1/2}\mathrm{r}^{-1/2}$ whose centers have distance at most R_k to some curve from Γ_k. The balls from $\mathfrak{U}_{k+}$ cover the set of points with distance R_k or less to some curve from Γ_k. The collection $\mathfrak{U}_{k+}$ has at most $c_0 c^3(\ln \mathrm{r})^4 m_k^{-3/2}\mathrm{r}^{1/2}$ balls.

The sets that comprise $\mathfrak{U}_N$ are balls of radius $\mathrm{r}^{-1/3}(\ln \mathrm{r})^{-c}$ with centers in $\mathcal{U}_N$. These sets define an open cover of $\mathcal{U}_N$. A cover of this sort can be found with less than $c_0(\ln \mathrm{r})^{c_0 c}$ elements, and such is the case for $\mathfrak{U}_N$.

Part 7: This part of the subsection defines the vector spaces $\{V_U\}_{U\in\mathfrak{U}}$. The next lemma is needed for the definition.

Lemma 7.11 *There exists $\kappa \geq 1$ with the following significance: Let $U \subset Y_Z$ denote a ball of radius $\rho \in (0, \kappa^{-1})$. Fix an isometric isomorphism between $E|_U$ and $U \times \mathbb{C}$. Use the latter to view the product connection on $U \times \mathbb{C}$ as a connection on $E|_U$. Use ∇_0 to denote the corresponding covariant derivative on $C^\infty(U; iT^*M \oplus \mathbb{S} \oplus i\mathbb{R})$. There exists a κ-dimensional vector space $W_U \in C^\infty(U; iT^*M \oplus \mathbb{S} \oplus i\mathbb{R})$ such that if $\mathfrak{q}$ is a section over U of $iT^*M \oplus \mathbb{S} \oplus i\mathbb{R}$ which is L^2-orthogonal to W_U, then $\int_U |\nabla_0\mathfrak{q}|^2 \geq \kappa^{-1}\rho^{-2}\int_U |\mathfrak{q}|^2$.*

This lemma will be proved momentarily; so assume it to be true for now.

Fix $U \subset \mathfrak{U}$. If $c \geq c_0$ then the radius of each ball from $\mathfrak{U}$ will be smaller than Lemma 7.12's version of κ^{-1} and each ball from $\mathfrak{U}$ will sit in the Gaussian coordinate chart about its center point. With this understood, fix $U \in \mathfrak{U}$ and let p denote U's center point. Fix an isometric isomorphism between $E|_p$ and $\mathbb{C}$ and use A's parallel transport along the radial geodesics from p to extend this identification to one between $E|_U$ and the product bundle $U \times \mathbb{C}$. Define V_U to be Lemma 7.11's vector space W_U.

Proof of Lemma B.11. If $\rho < c_0^{-1}$, then U has a Gaussian coordinate chart centered at its center point. Fix an isometric identification between K^{-1} at the center point of U with $\mathbb{C}$ and use the A_K parallel transport along the radial geodesics through the center point to extend this isomorphism to one between $K^{-1}|_U$ and $U \times \mathbb{C}$. Use the coordinate basis with the identification $K^{-1}|_U = U \times \mathbb{C}$ and the chosen identification $E|_U = U \times \mathbb{C}$ to give a product structure to T^*M and $\mathbb{S}$ over U. Having done so, rescale the coordinates so the ball of radius ρ becomes the ball of radius 1; then invoke the next lemma. □

Lemma 7.12 *Let $U \subset \mathbb{R}^3$ denote the ball of radius 1 centered on the origin. If $\mathfrak{h} \in C^\infty(U; \mathbb{C})$ is such that $\int_U \mathfrak{h} = 0$, then $\int_U |d\mathfrak{h}|^2 \geq \frac{1}{4}\int_U \mathfrak{h}^2$.*

Proof. It is sufficient to prove the bound for functions that depend only on z through its absolute value. This understood, use ρ to denote $|z|$ and let h denote a function that depends only on ρ and has integral zero over the unit ball. Let $h_* = h - h(1)$. Use integration by parts to see that

$$\int_0^1 \mathfrak{h}_*^2 \rho^{-1} d\rho \leq 2\int_0^1 |\frac{\partial}{\partial\rho}\mathfrak{h}|\, |\mathfrak{h}_*|\, \rho d\rho. \tag{7.41}$$

What is written in (7.41) implies that

$$\int_0^1 \mathfrak{h}_*^2 d\rho \leq 4\int_0^1 |d\mathfrak{h}|^2 \rho^2 d\rho. \tag{7.42}$$

Meanwhile, $\int_0^1 \mathfrak{h}_*^2 d\rho \geq \int_0^1 \mathfrak{h}_*^2 \rho^2 d\rho$, the latter being the integral of $\mathfrak{h}_*^2$ over the unit ball. This last integral is $\frac{1}{3}\mathfrak{h}(1)^2$ plus the integral of $\mathfrak{h}^2$ because the integral of $\mathfrak{h}$ is zero. □

Part 8: This step sets the stage for the specification of c and $\{\rho_k\}_{1\leq k\leq N-1}$ so as to guarantee (7.40). To start, let $U \subset \mathfrak{U}$ denote a given ball and let p denote the center point of U. Fix an isometric isomorphism between $E|_p$ and $\mathbb{C}$ and then use A's parallel transport along the radial geodesics from p to extend this isomorphism to give an isomorphism between $E|_U$ and $U \times \mathbb{C}$. Let θ_0 denote the product connection on $U \times \mathbb{C}$. Use the isomorphism just defined to view θ_0 as a connection on $E|_U$. Having done so, write A on U as $\theta_0 + \hat{a}_{A,U}$ with $\hat{a}_{A,U}$ being an $i\mathbb{R}$-valued 1-form on U. Let D_U denote the radius of U. The norm of $\hat{a}_{A,U}$ is bounded by $c_0\mathrm{D}_U \sup_U |B_A|$.

Fix $k \in \{1, \ldots, N-1\}$; let U denote a ball from either $\mathfrak{U}_{k-}$ or $\mathfrak{U}_{k0}$. It follows from what Lemma 7.2 that $|B_A| \leq c_0 m_k^2$ on U and so $|\,\hat{a}_{A,U}\,| \leq c_0 c^{-1}\rho_k m_k^2$. If $U \in \mathcal{U}_{k+}$, then it follows from Lemma 7.2 that $|B_A| \leq c_0 m_k^{-1}\mathrm{r}$ on U and so $|\,\hat{a}_{A,U}\,| \leq c_0 c^{-1} m_k^{-1/2}\mathrm{r}^{1/2}$ on U. If U is from $\mathfrak{U}_N$, then Lemma 7.3 finds $|B_A| \leq c_0 \mathrm{r}^{2/3}(\ln \mathrm{r})^{c_0}$ on U and so $|\,\hat{a}_{A,U}\,| \leq c_0 c^{-1}\mathrm{r}^{1/3}(\ln \mathrm{r})^{c_0}$.

Given $U \subset \mathfrak{U}$, use the isomorphism defined above between $E|_U$ and $U \times \mathbb{C}$ to again view θ_0 as a connection on $E|_U$. Use ∇_0 to denote the corresponding covariant derivative on $C^\infty(U; iT^*M \oplus \mathbb{S} \oplus V)$. Since $|\,\hat{a}_{A,U}\,|^2 \leq c_0 \sup_U |B_A|$ in all cases, so

$$|\nabla_A \mathfrak{q}|^2 \geq \frac{1}{2}|\nabla_0 \mathfrak{q}|^2 - c_0\,(\sup_U |B_A|)|\mathfrak{q}|^2 \tag{7.43}$$

for all $\mathfrak{q} \in C^\infty(U; iT^*M \oplus \mathbb{S} \oplus V)$.

Consider next the endomorphism $\mathcal{Q}$ that appears in (7.40). A look at the formula in (IV.A.12) finds

$$|\mathcal{Q}^-| \le c_0(1 + |B_A| + \mathrm{z}^{1/2}|\nabla_A\psi|) \quad \text{and} \quad |\mathcal{Q}^+| \ge c_0^{-1}\mathrm{z}|\psi|^2. \tag{7.44}$$

To say more about the bounds in (7.44) on the sets from $\mathfrak{U}$, fix first $k \in \{1, \ldots, N-1\}$ and let U denote a ball from $\mathfrak{U}_{k-}$ or $\mathfrak{U}_{k0}$. Lemma 7.2 finds $|\nabla_A\psi| \le c_0 m_k^{1/2}$ and $|\psi|^2 \ge c_0 m_k^{-1}$ on U. Since $|B_A|$ on U is bounded by $c_0 m_k^2$, the inequalities in (7.43) and (7.44) imply that

$$|\nabla_A\mathfrak{q}|^2 + \langle \mathfrak{q}, \mathcal{Q}^+\mathfrak{q}\rangle \ge |\nabla_0\mathfrak{q}|^2 + 2c_\diamond \sup_U |\mathcal{Q}^-|\,|\mathfrak{q}|^2 - c_0 m_k^2|\mathfrak{q}|^2 \tag{7.45}$$

for all $\mathfrak{q} \in C^\infty(U; iT^*M \oplus \mathbb{S} \oplus V)$. Meanwhile, if U is a ball from $\mathfrak{U}_{k+}$, then Lemma 7.3 finds $|\nabla_A\psi| \le c_0 m_k^{-1}\mathrm{r}^{1/2}$ and $|B_A| \le c_0 m_k^{-1}\mathrm{r}$. This being the case, then (7.43) and (7.44) find

$$|\nabla_A\mathfrak{q}|^2 + \langle \mathfrak{q}, \mathcal{Q}^+\mathfrak{q}\rangle \ge \frac{1}{2}|\nabla_0\mathfrak{q}|^2 + 2c_\diamond \sup_U |\mathcal{Q}^-|\,|\mathfrak{q}|^2 - c_0 m_k^{-1}\mathrm{r}|\mathfrak{q}|^2 \tag{7.46}$$

for all $\mathfrak{q} \in C^\infty(U; iT^*M \oplus \mathbb{S} \oplus V)$.

Suppose next that U is a ball from $\mathfrak{U}_N$. What is said in Lemma 7.3 implies that $|B_A| \le c_0\mathrm{r}^{2/3}(\ln \mathrm{r})^{c_0}$ and $|\nabla_A\psi| \le c_0\mathrm{r}^{1/6}$ on U, so (7.43) and (7.44) lead to the inequality

$$|\nabla_A\mathfrak{q}|^2 + \langle \mathfrak{q}, \mathcal{Q}^+\mathfrak{q}\rangle \ge \frac{1}{2}|\nabla_0\mathfrak{q}|^2 + 2c_\diamond \sup_U |\mathcal{Q}^-|\,|\mathfrak{q}|^2 - c_0\mathrm{r}^{2/3}(\ln \mathrm{r})^{c_0}|\mathfrak{q}|^2 \tag{7.47}$$

for all $\mathfrak{q} \in C^\infty(U; iT^*M \oplus \mathbb{S} \oplus V)$.

Part 9: This part of the subsection specifies c and $\{\rho_k\}_{1\le k\le N-1}$ so as to satisfy (7.40). To this end, suppose that $k \in \{1, \ldots, N-1\}$. Suppose that U is from $\mathfrak{U}_{k-}$ or $\mathfrak{U}_{k0}$. If $\mathfrak{q} \in C^\infty(U; iT^*M \oplus \mathbb{S} \oplus V)$ is L^2-orthogonal to the subspace V_U, then Lemma 7.11 and (7.45) find

$$\int_U \big(|\nabla_A\mathfrak{q}|^2 + \langle \mathfrak{q}, \mathcal{Q}^+\mathfrak{q}\rangle\big) \ge (c_0^{-1}\rho_k^{-2} - c_0 m_k^2 + 2c_\diamond \sup_U |\mathcal{Q}^-|) \int_U |\mathfrak{q}|^2. \tag{7.48}$$

It follows as a consequence that (7.40) holds if $\rho_k^{-2} \ge c_0(T^2 + m_k^2)$ and this is so if $c > c_0$. Suppose next that U is from $\mathfrak{U}_{k+}$ and that $\mathfrak{q} \in C^\infty(U; iT^*M \oplus \mathbb{S} \oplus V)$ is L^2-orthogonal to V_U. Lemma 7.11 and (7.46) imply that

$$\int_U \big(|\nabla_A\mathfrak{q}|^2 + \langle \mathfrak{q}, \mathcal{Q}^+\mathfrak{q}\rangle\big) \ge \big((c_0^{-1}c^2 - c_0)m_k^{-1}\mathrm{r} + 2c_\diamond \sup_U |\mathcal{Q}^-|\big) \int_U |\mathfrak{q}|^2, \tag{7.49}$$

if q is L^2 -orthogonal to V_U. Thus (7.40) holds if $c \geq c_0(1 + m_k \mathrm{r}^{-1} T^2)$; and in particular, (7.40) holds for $c > c_0$ if the eigenvalue bound T is less than $\mathrm{r}^{1/6}(\ln \mathrm{r})^{-c_0}$.

The last case to consider is that where U comes from $\mathfrak{U}_N$. Lemma 7.11 and (7.47) imply for such U that

$$\int_U \big(|\nabla_A \mathfrak{q}|^2 + \langle \mathfrak{q}, \mathcal{Q}^+ \mathfrak{q} \rangle\big) \geq \big((c_0^{-1}(\ln \mathrm{r})^{2c} - (\ln \mathrm{r})^{c_0} \mathrm{r}^{2/3} + 2c_\diamond \sup_U |\mathcal{Q}^-|\big) \int_U |\mathfrak{q}|^2 \tag{7.50}$$

if $\mathfrak{q}$ is L^2-orthogonal to V. It follows as a consequence that (7.40) holds for such U if both $c > c_0$ and the eigenvalue bound T is less than $\mathrm{r}^{1/3}$.

Granted all of the above, and given that $T < \mathrm{r}^{1/6}(\ln \mathrm{r})^{-c}$, then (7.40) holds for all sets from $\mathfrak{U}$ if $c > c_0$. This understood, choose c to be twice this lower bound.

Part 10: The dimension of each $U \in \mathfrak{U}$ version of V_U is bounded by c_0, and so it follows from what is said at the end of Part 5 that $\mathfrak{n}_T$ is no greater than c_0 times the number of sets in the collection $\mathfrak{U}$.

An upper bound for size of $\mathfrak{U}$ is obtained by summing upper bounds for the sizes of the various $k \in \{1, \ldots, N\}$ versions of $\mathfrak{U}_k$. Let N_T denote the largest value of k such that $T > m_k$ and suppose first that $k \in \{1, \ldots, N_T\}$. It follows from what is said in Part 6 that $\mathfrak{U}_{k-}$ contains no more than $c_0 T^3 m_k^{-3}$ sets. Meanwhile, $\mathfrak{U}_{k0}$ and $\mathfrak{U}_{k+}$ together contain at most $c_0 m_k^{-3/2} \mathrm{r}^{1/2}(\ln \mathrm{r})^{c_0}$ balls. Thus $\bigcup_{1 \leq k \leq N_T} \mathfrak{U}_k$ contains at most $c_0(T^3 + \mathrm{r}^{1/2}(\ln \mathrm{r})^{c_0})$ balls. Suppose next that $k \in \{N_T + 1, \ldots, N - 1\}$. In this case, $\mathfrak{U}_{k-}$ has at most c_0 balls while $\mathfrak{U}_{k0}$ and $\mathfrak{U}_{k+}$ again have at most $c_0 m_k^{-3/2} \mathrm{r}^{1/2}(\ln \mathrm{r})^{c_0}$ balls. Thus, $\bigcup_{N_T < k \leq N-1} \mathfrak{U}_k$ contains at most $c_0 T^{-3/2} \mathrm{r}^{1/2}(\ln \mathrm{r})^{c_0}$ balls. As noted in Part 6, the set $\mathfrak{U}_N$ has at most $c_0(\ln \mathrm{r})^{c_0}$ balls.

Given that $T \leq c_0 \mathrm{r}^{1/6}(\ln \mathrm{r})^{c_0}$, the bounds just stated imply that $\mathfrak{n}_T \leq c_0 \mathrm{r}^{1/2}(\ln \mathrm{r})^{c_0}$. Therefore, (7.37) bounds the spectral flow along the families $\{\mathcal{L}_{2,s}\}_{s \in [0,1]}$ and $\{\mathcal{L}_{3,s}\}_{s \in [0,1]}$ by $c_0 T^{-1} \mathrm{r}(\ln \mathrm{r})^{c_0}$. This understood, take $T = \mathrm{r}^{1/7}(\ln \mathrm{r})^{c_0}$ to obtain the bound in (7.33). □

7.8 The proof of Proposition 3.6

If Y_Z has a single component, then the function $\mathfrak{f}_s$ is defined in Section 7.6. Proposition 3.6's assertion in this case is implied directly by Lemma 7.9's fourth bullet.

Suppose now that Y_Z has more than 1 component. To define $\mathfrak{f}_s$ in this case, introduce $\mathcal{Y}$ to denote the set of components of Y_Z. The space $\mathrm{Conn}(E) \times C^\infty(Y_Z; \mathbb{S})$ can be

written as $\prod_{Y'\in\mathcal{Y}}(\mathrm{Conn}(E|_{Y'}) \times C^\infty(Y';\mathbb{S}|_{Y'}))$. Section 7.6 defines any given $Y' \in \mathcal{Y}$ version of $\mathfrak{f}_s$ on $\mathrm{Conn}(E|_{Y'}) \times C^\infty(Y';\mathbb{S}|_{Y'})$. Denote the latter by $\mathfrak{f}_{s;Y'}$. Set

$$\mathfrak{f}_s = \sum_{Y'\in\mathcal{Y}} \mathfrak{f}_{s;Y'}.$$

Each $Y' \in \mathcal{Y}$ has its version of the function $\mathfrak{a}$ on $\mathrm{Conn}(E|_{\mathcal{Y}'}) \times C^\infty(Y';\mathbb{S}|_{Y'})$. Use $\mathfrak{a}_{Y'}$ to denote the latter. Then $\mathfrak{a}^{\mathfrak{f}} = \sum_{Y'}(\mathfrak{a}_{Y'} + 2\pi(\mathrm{r}-\pi)\mathfrak{f}_{s;Y'})$. This understood, it is enough to bound $|\mathfrak{a}_{Y'} + 2\pi(\mathrm{r}-\pi)\mathfrak{f}_{s;Y'}|$ for each $Y' \in \mathcal{Y}$. Lemma 7.9 supplies a suitable bound when $c_1(\det(\mathbb{S}|_{Y'}))$ is not torsion. This understood, suppose $Y' \in \mathcal{Y}$ and $c_1(\det(\mathbb{S})|_{Y'})$ is torsion. Thus, $w = 0$ on Y'.

Write ψ on Y' as $\mathrm{r}^{-1/2}\lambda$ to see that the set of solutions to (2.4) on Y' is r-independent. It follows as a consequence of what is said in Chapter 5 of [KM] that the space of $C^\infty(Y';S^1)$-orbits of solutions to (2.4) on Y' is compact. Hold on to this fact for the moment. Write ψ in the Y' version of (7.30) as $\mathrm{r}^{1/2}\lambda$ and write the sections b and ϕ as $(\mathrm{r}z)^{1/2}b'$ and $(\mathrm{r}z)^{1/2}\phi'$ to see that the spectrum of the operator in (7.30) depends neither on r nor z. What was just said about compactness and what was just said about the spectrum implies directly that $|\mathfrak{a}_{Y'} + 2\pi(\mathrm{r}-\pi)\mathfrak{f}_{s;Y'}| \le c_0$.

8 Cobordisms and the Seiberg-Witten equations

This section proves Propositions 3.4 and 3.7. Section 8.1 states three key lemmas that are used in Section 8.2 to prove Proposition 3.4. Sections 8.5-8.6 prove the three lemmas. Section 8.7 contains the proof of Proposition 3.7.

8.1 The three key lemmas

The three parts of this subsection supply three lemmas that assert pointwise bounds for ψ, the curvature of A and for the covariant derivative of ψ. These bounds are used in the next subsection to prove Proposition 3.3. All three lemmas assume implicitly that the conditions in Section 3.3 are satisfied. Additional assumptions are stated when needed.

Part 1: The first lemma starts the story with a pointwise bound for $|\psi|$ and L^2-bounds on F_A and the covariant derivatives of ψ. With regards to notation, this lemma uses $(\nabla_{\mathbb{A}}\psi)_s$ to denote the section of $\mathbb{S}^+$ over the $|s| > 1$ part of X that gives the pairing between $\nabla_{\mathbb{A}}\psi$ and the vector field $\frac{\partial}{\partial s}$.

Lemma 8.1 *There exists $\kappa > 1$ such that given any $c \geq \kappa$, there exists κ_c with the following significance: Fix $\mathrm{r} \geq \kappa_c$. If X is not the product cobordism, assume that the metric obeys (2.8) with $L \leq c$, that the norm of the Riemann curvature is bounded by $\mathrm{r}^{1/c}$ and that the norm of w_X is bounded by c. Fix μ_- and μ_+ from the Y_- and Y_+ versions of Ω with $\mathcal{P}$-norm bounded by 1 and use this data to define the equations in (2.9). Suppose that $\mathfrak{d} = (\mathbb{A}, \psi)$ is an instanton solution to these equations. Then $|\psi| \leq \kappa_c$. If X is not the product cobordism, assume in addition that the volume of the s-inverse image of any length 1 interval is bounded by c and that the metric's injectivity radius is greater than $\mathrm{r}^{-1/c}$. Also assume in this case that $L_{\mathrm{tor}} \leq c\mathrm{r}$ and that w_X obeys (2.11) plus Item c) of the fourth bullet of (3.13). Let $\mathfrak{c}_-$ and $\mathfrak{c}_+$ denote the respective $s \to -\infty$ and $s \to \infty$ limits of $\mathfrak{d}$ and suppose that $\mathfrak{a}(\mathfrak{c}_-) - \mathfrak{a}(\mathfrak{c}_+) \leq c\mathrm{r}^2$. Then*

- *The L^2-norms of $|F_{\mathbb{A}}(\frac{\partial}{\partial s}, \cdot)|$ and $\mathrm{r}^{1/2}|(\nabla_{\mathbb{A}}\psi)_s|$ on the $|s| \geq L$ part of X are less than $\kappa_c \mathrm{r}$.*
- *The L^2-norms of $F_{\mathbb{A}}$ and $\mathrm{r}^{1/2}\nabla_{\mathbb{A}}\psi$ on the s-inverse image of any length 1 interval in $\mathbb{R}$ are no greater than $\kappa_c \mathrm{r}$.*

This lemma is proved in Section 8.3.

Part 2: The next lemma supplies a refined set of bounds for $|\alpha|$ and its covariant derivatives on U_C and U_0. This lemma and the subsequent lemma implicitly write $\mathbb{S}^+$ on U_C and U_0 as $E \oplus (E \otimes K^{-1})$. Having done so, they then write ψ with respect to this splitting as (α, β); and they write the connection $\mathbb{A}$ as $\mathbb{A} = A_K + 2A$ with A being a connection on E.

The notation in these upcoming lemmas refers to the complex structure on U_C and U_0 that is defined using the metric and the compatible symplectic form $\mathrm{ds} \wedge *w + w$. The $(1,0)$-part of the complexified cotangent space for this complex structure is the direct sum of the span of $ds + i * w$ and dz on U_C and it is the direct sum of the span of $ds + i * w$ and the $(1,0)$-part of the tangent space to the constant-(s,u) spheres in U_0 with the complex structure on S^2 being the standard one. These lemmas write $\nabla_A\alpha$ with respect to the $(1,0)$- and $(0,1)$-splitting of the complexified cotangent bundle as $\partial_A\alpha + \bar\partial_A\alpha$ with $\partial_A\alpha$ denoting $(1,0)$-part of $\nabla_A\alpha$ and with $\bar\partial_A\alpha$ denoting the $(0,1)$-part. The lemma and the subsequent also introduce ρ_D to denote the diameter of the cross-sectional disk D that is used to define U_C.

Lemma 8.2 *There exists $\kappa > 100\,(1 + \rho_D^{-1})$ such that given any $c \geq \kappa$, there exists $\kappa_c \geq \kappa$ with the following significance: Fix $\mathrm{r} \geq \kappa_c$ and assume that the metric obeys (2.8), (3.12), and the $(c, r = \mathrm{r})$-versions of the conditions in the first two bullets of*

(3.13). Assume that $|w_X| \le c$ and that w_X obeys (3.11). Fix elements μ_- and μ_+ from the Y_- and Y_+ versions of Ω with $\mathcal{P}$-norm bounded by 1. Assume in addition that their norms and those of their derivatives to order 10 on U_γ and $\mathcal{H}_0$ are bounded by $e^{-\mathrm{r}^2}$. Use this data to define the equations in (2.9). Let $\mathfrak{c}_-$ and $\mathfrak{c}_+$ denote respective solutions to the (r, μ_-)-version of (2.4) on Y_- and the (r, μ_+)-version of (2.4) with $\mathfrak{a}(\mathfrak{c}_-) - \mathfrak{a}(\mathfrak{c}_+) \le c\mathrm{r}^2$, and suppose that $\mathfrak{d} = (\mathbb{A}, \psi)$ is an instanton solution to (2.9) with $s \to -\infty$ limit equal to $\mathfrak{c}_-$ and $s \to \infty$ limit equal to $\mathfrak{c}_+$. If p is a point in one of the domains U_C or U_0 with distance greater than $\kappa^2 \mathrm{r}^{-1/2}(\ln \mathrm{r})^2$ from the domain's boundary, then the following holds at p:

- $|\beta|^2 \le e^{-\sqrt{\mathrm{r}}/\kappa^2}$ *and* $|\alpha|^2 \le 1 + e^{-\sqrt{\mathrm{r}}/\kappa^2}$.
- $|\nabla_A \beta| + |\nabla_A \nabla_A \beta| \le e^{-\sqrt{\mathrm{r}}/\kappa^2}$.
- $|\bar{\partial}_A \alpha| \le e^{-\sqrt{\mathrm{r}}/\kappa^2}$.
- *If $|\alpha|^2 \in (\kappa^{-1}, 1 - \kappa^{-1})$ at p, then either $|\nabla_A \alpha|^2 \ge \kappa^{-3}\mathrm{r}$ at p or the Hessian $\nabla d|\alpha|^2$ at p has an eigenvalue with absolute value greater than $\kappa^{-3}\mathrm{r}$.*
- $|\nabla_A \alpha| + \mathrm{r}^{-1/2}|\nabla_A(\nabla_A \alpha)| \le \kappa \mathrm{r}^{1/2}$ *if $|F_A| \le c\mathrm{r}$ on the radius $\kappa \mathrm{r}^{-1/2}$-ball centered at p.*

This lemma is proved in Section 8.4.

Part 3: The final lemma here writes F_A on U_C and U_0 as $F_A = ds \wedge \mathcal{E}_A + *\mathcal{B}_A$ with $\mathcal{E}_A$ and $\mathcal{B}_A$ denoting s-dependent, $i\mathbb{R}$ valued 1-forms on either $\mathbb{R}/(\ell_\gamma \mathbb{Z}) \times D$ or $\mathcal{H}_0$ as the case may be. These 1-forms are written as

$$
\begin{cases} \mathcal{E}_A = -i(1-\sigma)(\mathrm{r}(1-|\alpha|^2) + \mathfrak{z}_A)\, dt + \mathfrak{r} + \mathfrak{X} \quad \text{and} \\ \mathcal{B}_A = -i\sigma(\mathrm{r}(1-|\alpha|^2) + \mathfrak{z}_B)\, dt + \mathfrak{r} - \mathfrak{X}, \end{cases} \tag{8.1}
$$

where $\mathfrak{z}_A$ and $\mathfrak{z}_B$ are functions, and where both $\mathfrak{r}$ and $\mathfrak{X}$ annihilate the vector field $\frac{\partial}{\partial t}$.

Lemma 8.3 *There exists $\kappa > \pi$ such that given any $c \ge \kappa$, there exists $\kappa_c > 200(1 + \rho_D^{-1})$ with the following significance: Fix $\mathrm{r} \ge \kappa_c$ and assume that the metric and w_X are $(c, r = \mathrm{r})$-compatible. Fix elements μ_- and μ_+ from the Y_-- and Y_+-versions of Ω with $\mathcal{P}$-norm bounded by 1. Assume in addition that their norms and those of their derivatives up to order 10 on U_γ and $\mathcal{H}_0$ are bounded by $e^{-\mathrm{r}^2}$. Use all of these data to define the equations in (2.9). Let $\mathfrak{c}_-$ and $\mathfrak{c}_+$ denote the respective solutions to the (r, μ_-)-version of (2.4) on Y_- and the (r, μ_+) version of (2.4) on Y_+ with $\mathfrak{a}(\mathfrak{c}_-) - \mathfrak{a}(\mathfrak{c}_+) \le \mathrm{r}^{2-1/c}$. Suppose that $\mathfrak{d} = (\mathbb{A}, \psi)$ is an instanton solution to (2.9) with $s \to -\infty$ limit equal to $\mathfrak{c}_-$ and $s \to \infty$ limit equal to $\mathfrak{c}_+$. Let p denote a point*

in either one of the domains U_C or U_0 with distance κ^{-1} or more from the domain's boundary. Then the following are true at p:

- $-\mathrm{r}^{-100} < 1-\sigma < 1+\mathrm{r}^{-100}$.
- $|\mathfrak{z}_A| + |\mathfrak{z}_B| \le \mathrm{r}^{-100}$.
- $|\mathfrak{r}| \le \kappa\mathrm{r}^{-100}$.
- $|\mathfrak{X}|^2 \le 2\mathrm{r}^2\sigma(1-\sigma)(1-|\alpha|^2) + \kappa\mathrm{r}^{-100}$.
- $|\nabla\mathcal{E}_A| + |\nabla\mathcal{B}_A| \le \kappa\mathrm{r}^{3/2}$.

Lemma 8.3 is proved in Section 8.5 modulo a key lemma which is proved in Section 8.6.

8.2 Proof of Proposition 3.4

This part of the subsection uses what is said in Lemmas 8.1-8.3 to prove Proposition 3.4. The argument assumes that the integral of $iF_{\hat{A}}$ over C is negative so as to derive nonsense. This is done in the seven parts that follow. Before starting, note that the assumptions in this proposition allow Lemmas 8.1 and 8.3 to be invoked, and the conclusions of Lemma 8.3 imply in particular that Lemma 8.2 can be invoked as well.

Part 1: This first part of the proof sets the stage for what is to come by supplying two observations about the pull-back of $iF_{\hat{A}}$ to C. What follows is the first observation:

(8.2) The integral of $\dfrac{i}{2\pi}F_{\hat{A}}$ over C is an integer.

This follows from Lemma 7.6 since the latter implies that $\hat{A}$ is flat and $\alpha/|\alpha|$ is $\hat{A}$-covariantly constant where $|s| \gg 1$ on C.

The second observation concerns the function F on C that is defined by writing the pull-back to C of $iF_{\hat{A}}$ as $\mathrm{F}\, ds \wedge dt$:

(8.3) The function F is nearly non-negative in the sense that $\mathrm{F} \ge -c_0\mathrm{r}^{-100}$.

This follows directly from the formula given below for F using the second bullet of Lemma 8.2 and the first and second bullets of Lemma 8.3. The upcoming formula for F uses $(\partial_A\alpha)_0$ to denote the $ds + i*w$ component of $\partial_A\alpha$ and use $(\bar{\partial}_A\alpha)_0$ to denote the $ds - i*w$ component of $\bar{\partial}_A\alpha$. Here is the promised formula for F:

(8.4) $$\mathrm{F} = (1-\wp)(1-\sigma)(\mathrm{r}(1-|\alpha|^2) + \mathfrak{z}_A) + \wp'(|(\partial_A\alpha)_0|^2 - |(\bar{\partial}_A\alpha)_0|^2).$$

This formula follows directly from (3.15) and (8.1).

Part 2: Let $\mathbb{I} \subset \mathbb{R}$ denote the set characterized as follows: A point s is in $\mathbb{I}$ if the the integral of F over the slice $\{s\} \times \gamma$ in C is negative. The following assertion is a direct consequence of (8.2) and (8.3):

(8.5) If $\int_C iF_{\hat{A}} < 0$ then the measure of the set $\mathbb{I}$ is greater than $c_0^{-1}\mathrm{r}^{100}$.

Granted (8.5), there are at least $c_0^{-1}\mathrm{r}^{100}$ disjoint open intervals of length 1 in $\mathbb{R}$ with center point in $\mathbb{I}$. This understood, use the first bullet of Lemma 8.1 to find an interval $I \subset \mathbb{R}$ of length 1 with center point in $\mathbb{I}$, with $|s| > L+2$ and such that

$$(8.6) \qquad \int_{I\times Y} \big(|F_A(\frac{\partial}{\partial s}, \cdot)|^2 + \mathrm{r}|(\nabla_A \psi)_s|^2\big) < \mathrm{r}^{-97}.$$

This inequality enters the story in Part 6.

Part 3: Let $s \in I$ denote I's center point and thus a point for which the integral of F over $\{s\} \times \gamma$ is negative. As will be explained momentarily, the lower bound in (8.3) for F leads to the following observation:

(8.7) The variation of $\wp$ over $\{s\} \times \gamma$ is no greater than $c_0\mathrm{r}^{-50}$.

To prove this, first use the fundamental theorem of calculus to see that

$$(8.8) \qquad \sup_{\{s\}\times\gamma} \wp - \inf_{\{s\}\times\gamma} \wp \le c_0 \int_{\{s\}\times\gamma} \wp'|(\nabla_A\alpha)_0|^2.$$

The bound in (8.7) follows from (8.8) using the lower bound for F and the third bullet of Lemma 8.2. It follows from (8.8) that $|\alpha|^2 \le \frac{5}{8}$ on the whole of γ.

Part 4: This part uses the conclusions of Part 3 to deduce the following:

(8.9) The function σ on the $|u| < 1$ part of $\{s\} \times (\gamma \cap \mathcal{H}_0)$ obeys $\sigma < c_0\mathrm{r}^{-33}$.

To see why this is the case, let (s,p) denote a given point in the $|u| < 1$ part of $\{s\}\times(\gamma\cap\mathcal{H}_0)$ where $\sigma > 0$. Let S denote the cross-sectional sphere in $\mathcal{H}_0$ that contains p. Use (3.15) to write the pull-back of F_A to S as $\frac{1}{2}\mathrm{B}dz\wedge d\bar{z}$ with $\mathrm{B} = \sigma(\mathrm{r}(1-|\alpha|^2)+\mathfrak{z}_B)$. Use ε to denote the value of σ at (s,p). Invoke the first and second bullets of Lemma 8.3 to conclude that value of B at (s,p) is greater than $\frac{5}{8}\mathrm{r}\varepsilon - c_0\mathrm{r}^{-100}$. The fifth bullet of Lemma 8.3 finds that $\mathrm{B} > c_0^{-1}\mathrm{r}\varepsilon$ on the radius $c_0^{-1}\mathrm{r}^{1/2}\varepsilon$ disk in the cross-sectional sphere $\{s\} \times S$ with center at (s,p). Meanwhile, the first bullets of Lemma 8.3 and Lemma 8.2 imply that $\mathrm{B} > -c_0\mathrm{r}^{-99}$ on the whole of $\{s\} \times S$, and so the integral of B over $\{s\} \times S$ is no less than $c_0^{-1}\varepsilon^3 - c_0\mathrm{r}^{-99}$. This integral must be zero because the first Chern class of E has zero pairing with the cross-sectional spheres in $\mathcal{H}_0$. Thus $\varepsilon \le c_0\mathrm{r}^{-33}$.

Part 5: What is said in Part 4 implies that $(1-\wp) < c_0 \mathrm{r}^{-50}$ on $\{s\}\times\gamma$. Indeed, if this bound is violated, then it follows from (8.7) and the formula for F in (8.4) that the integral of F over the $|u| < 1$ part of $\{s\}\times\gamma$ is greater than $c_0\mathrm{r}^{-49}$. Given the lower bound in (8.3), this last lower bound runs afoul of the assumption that F's integral over $\{s\}\times\gamma$ is negative. The small size of $1-\wp$ implies in particular that $|\alpha|^2 > \frac{3}{8}$ on $\{s\}\times\gamma$.

Part 6: Granted the conclusions of Parts 4 and 5, then the fourth bullet of Lemma 8.2 asserts that one or the other of the following are true at each point in the $|u| < 1$ part of $\{s\}\times(\gamma\cap\mathcal{H}_0)$: Either $|\nabla_A\alpha|^2 \geq c_0^{-1}\mathrm{r}$ or the Hessian matrix $\nabla d|\alpha|^2$ has an eigenvalue with absolute value greater than $c_0^{-1}\mathrm{r}$. As explained next, this has the following consequence:

(8.10) Let $(\partial_A\alpha)_1$ denote the component of $\partial_A\alpha$ that annihilates both $\frac{\partial}{\partial s}$ and the kernel of w. Then $|(\partial_A\alpha)_1|^2$ is greater than $c_0^{-1}\mathrm{r}^{1/2}$ at all points in a radius $c_0^{-1}\mathrm{r}^{-1/2}$ ball with center at distance less than $c_0\mathrm{r}^{-1/2}$ from each point in the $|u| < 1$ part of $\{s\}\times(\gamma\cap\mathcal{H}_0)$.

To prove this, suppose first that $|\nabla_A\alpha|^2 \geq c_0^{-1}\mathrm{r}$ at a given point. Use the third bullet of Lemma 8.2 to see that one or both of $|(\partial_A\alpha)_1|^2$ and $|(\partial_A\alpha)_0|^2$ are greater than $c_0^{-1}\mathrm{r}$. In the latter case, the third bullet of Lemma 8.2 implies that $|(\nabla_A\alpha)_s|^2$ is greater than $c_0^{-1}\mathrm{r}$ at the point, and the second derivative bound from the fifth bullet of Lemma 8.2 implies that $|(\nabla_A\alpha)_s|^2 \geq c_0^{-1}\mathrm{r}$ at all points in a radius $c_0^{-1}\mathrm{r}^{-1/2}$ ball centered on this point. This being the case, the integral of $|(\nabla_A\alpha)_s|^2$ over this ball is greater than $c_0^{-1}\mathrm{r}^{-1}$ and this violates (8.6). Granted that $|(\nabla_A\alpha)_1|^2 \geq c_0^{-1}\mathrm{r}$ at the given point, then the second derivative bound from the fifth bullet of Lemma 8.2 implies what is asserted by (8.10).

Now suppose that the Hessian matrix $\nabla d|\alpha|^2$ at the given point has an eigenvalue that is greater than $c_0^{-1}\mathrm{r}$. Let v denote a unit length eigenvector at the point with such an eigenvalue. As will be explained directly, this vector must be such that $|ds(v)| + |dt(v)| < \frac{1}{100}$. To see why this is the case, suppose to the contrary that the latter bound is violated at a given point. It then follows from the first and fifth bullets of Lemma 8.2 that $|(\nabla_A\alpha)_s| \geq c_0^{-1}\mathrm{r}^{1/2}$ at all points in some ball of radius $c_0^{-1}\mathrm{r}^{-1/2}$ whose center has distance at most $c_0\mathrm{r}^{1/2}$ from the given point. This implies in particular that the integral of $|(\nabla_A\alpha)_s|^2$ over this same ball is no less than $c_0\mathrm{r}^{-1}$. But this is nonsense as it runs afoul of (8.6).

The fact that v is a unit length vector implies that $|dz(v)| > \frac{1}{2}$. Use this lower bound for $|dz(v)|$ with the third bullet of Lemma 8.2 and the second derivative bounds from

the fifth bullet of Lemma 8.2 to see that $|(\partial_A\alpha)_1|^2 \geq c_0^{-1}\mathrm{r}^{1/2}$ at all points in a ball of radius $c_0^{-1}\mathrm{r}^{-1/2}$ whose center point has distance at most $c_0\mathrm{r}^{-1/2}$ from the given point.

Part 7: Introduce the connection $\hat{A}_1$ on E's restriction to $I \times \mathcal{H}_0$ that is obtained from (A, α) by the formula $\hat{A}_1 = A - \frac{1}{2}(\bar{\alpha}\nabla_A\alpha - \alpha\nabla_A\bar{\alpha})$. The curvature 2-form of $\hat{A}_1$ is

$$F_{\hat{A}_1} = (1 - |\alpha|^2)F_A + \nabla_A\alpha \wedge \nabla_A\bar{\alpha}. \tag{8.11}$$

Let (s', p') denote the center point of a ball that is described by (8.10). Introduce $S \subset \mathcal{H}_0$ to denote the cross-sectional sphere that contains the point p'. Use (3.15) to write the pull-back of the curvature of $\hat{A}_1$ to $\{s\} \times S$ as $\mathrm{B}_1 dz \wedge d\bar{z}$ with B_1 given by

$$\mathrm{B}_1 = \sigma(1 - |\alpha|^2)(\mathrm{r}(1 - |\alpha|^2) + \mathfrak{z}_A) + |(\partial_A\alpha)_1|^2 - |(\bar{\partial}_A\alpha)_1|^2, \tag{8.12}$$

with $(\bar{\partial}_A\alpha)_1$ denoting here the $d\bar{z}$ component of $\bar{\partial}_A\alpha$. The function B_1 is also very nearly non-negative in the sense that $\mathrm{B}_1 \geq -c_0\mathrm{r}^{-100}$, this being a consequence of what is said in the first and third bullets of Lemma 8.2 and the first and second bullets of Lemma 8.3. This understood, then it follows from (8.10) and this lower bound for B_1 that the integral of B_1 over $\{s'\} \times S$ is positive. But this is nonsense because the latter integral computes 2π times the pairing of the first Chern class of E with the homology class defined by S, and this pairing is equal to zero. □

8.3 Proof of Lemma 8.1

The bounds in the lemma constitute a particular case of bounds that are used in Chapter 24 of [KM]. As most of the machinery behind what is done in [KM] is not needed for the proofs, the argument for Lemma 8.1 is presented momentarily. What follows directly lays a convention that is invoked implicitly in the arguments for Lemma 8.1 and in some of the subsequent lemmas.

If X is the product $\mathbb{R} \times Y_Z$, the bundles E and K^{-1} over Y_Z pull back via the projection to define bundles over X; their connections A_E and A_K likewise pull back to define connections on these bundles. The bundle $\det(\mathbb{S}^+)$ is isomorphic to $E^2 \otimes K^{-1}$ and thus to the pull-back of $\det(\mathbb{S})$. Fix once and for all an isometric isomorphism.

Suppose now that X is not a product. Use the embedding in the second bullet of (2.7) to identify the $s \leq -1$ part of X with $(-\infty, -1] \times Y_-$, and then use the projection to Y_- to view the Y_- version of the bundle $\mathbb{S}$ as bundles over the $s \leq -1$ part of X. The bundles $\mathbb{S}^+$ and $\mathbb{S}^-$ are isometrically isomorphic to $\mathbb{S}$ via an isomorphism

that covers the isomorphisms between both Λ^+ and Λ^- and T^*Y given by the interior product with $\frac{\partial}{\partial s}$. Fix such an isomorphism once and for all. This induces a Hermitian isomorphism between the bundle $\det(\mathbb{S}^+)$ over the $s < -1$ part of X and the Y_- version $\det(\mathbb{S})$. Fix once and for all an isometric isomorphism between these bundles. Use this isomorphism with the pull back via the composition of the embedding from (2.7)'s second bullet and the projection to Y_- to view $A_K + 2A_E$ as a Hermitian connection on the $s \leq -1$ part of $\det(\mathbb{S}^+)$. The analogous constructions can be made on the $s > 1$ part of X using the Y_+ version of $\mathbb{S}$ and so define an incarnation of the Y_+ version of $A_K + 2A_E$ as a Hermitian connection on $\det(\mathbb{S}^+)$.

Suppose for the moment that $\mathbb{A}$ is a given Hermitian connection on $\det(\mathbb{S}^+) \to X$. If X is the product $\mathbb{R} \times Y_Z$, then $\mathbb{A}$ can be written as $A_K + 2A$ with A now a connection on the bundle $E \to X$. There is a map $\hat{u} : X \to S^1$ such that $A - \hat{u}^{-1} d\hat{u} = A_E + \hat{a}_A$ where $\hat{a}_A$ annihilates the vector field $\frac{\partial}{\partial s}$. If X is not the product, then $\mathbb{A}$ can be written as $A_K + 2A$ on the $s \leq -1$ and $s \geq 1$ parts of X with A being a connection on the incarnation of E over the relevant part of X In this case, there exists a map $\hat{u}$ as just described but with domain the $s \leq -1$ part of X, and likewise there exists such a map with domain the $s \geq 1$ part of X.

The map $\hat{u}$ in the case when $X = \mathbb{R} \times Y_Z$ is unique up to multiplication by an s-independent map from Y_Z to S^1, and in the other cases, it is unique up to a map from the either the $s \leq -1$ or $s \geq 1$ part of X whose differential annihilates $\frac{\partial}{\partial s}$. The convention in each case is to take a map $\hat{u}$ whose restrictions to the constant s slices of its domain are homotopic to the constant map to S^1.

The connection $A_* = A - \hat{u}^{-1} d\hat{u}$ can be viewed as a map from $\mathbb{R}$ or $(-\infty, -1]$ or $[1, \infty)$ to $\mathrm{Conn}(E|_{Y_*})$ with Y_* either Y_Z or Y_- or Y_+ as the case may be. If ψ is a given section over X of $\mathbb{S}$, then $\psi_* = \hat{u}\psi$ can likewise be viewed as a map from $\mathbb{R}$ or $(-\infty, -1]$ or $[1, \infty)$ to $C^\infty(Y_*; \mathbb{S}|_{Y_*})$. When viewed in this light, the equations in (2.9) can be written as equations for (A_*, ψ_*) on the whole of X when X is the product cobordism, and on the $s \leq -L$ and $s \geq L$ parts of X when X is not the product cobordism. These equations are:

$$
\text{(8.13)} \qquad \begin{cases} \frac{\partial}{\partial s} A_* + B_{A_*} - \mathrm{r}\,(\psi_*^\dagger \tau \psi_* - i * w_{X_*}) - \frac{1}{2} B_{A_K} - i d\mu_* = 0 \quad \text{and} \\ \frac{\partial}{\partial s} \psi_* + D_{A_*} \psi_* = 0. \end{cases}
$$

The notation here uses w_{X*} to denote the 2-form w when X is the product cobordism. When X is not the product cobordism, w_{X*} denotes the s-dependent 2-form that is defined on the relevant constant s slices of X by the pull-back of w_X. In particular, $w_{X*} = w$ on the components of the $s \leq -L$ and $s > L$ parts of X where $c_1(\det(\mathbb{S}))$ is

not torsion. What is denoted in (8.13) by μ_* is either μ, μ_-, or μ_+ as the case may be.

Proof. The proof has four steps.

Step 1: The assertion that $|\psi| \le \kappa$ is proved by using the Weitzenböck formula to write $\mathcal{D}^-_{\mathbb{A}}\mathcal{D}^+_{\mathbb{A}}\psi$ as $\nabla^\dagger_{\mathbb{A}}\nabla_{\mathbb{A}}\psi + \mathrm{cl}(F^+_{\mathbb{A}})\psi + \frac{1}{4}\mathrm{R}\psi$ where R denotes the scalar curvature of the Riemannian metric. Granted this rewriting, it then follows from (2.9) and from the assumed bound on the norm of Riemann curvature that the function $|\psi|$ obeys the differential inequality $d^\dagger d|\psi| + \mathrm{r}(|\psi|^2 - |w_X| - c_c)|\psi| \le 0$. Use the maximum principle with this last inequality and the large $|s|$ bounds on $|\psi|$ that follow from Lemma 7.1 to see that $|\psi| \le c + c_0$.

Step 2: Let L_* denote either L or L_{tor}. Then use $\mathbb{I} \subset \mathbb{R}$ to denote either $\mathbb{R}$, $(-\infty, -L]$ or $[L, \infty)$. Define Y_* to be Y_Z in the case when $\mathbb{I} = \mathbb{R}$. When $\mathbb{I} = (-\infty, L_*]$ or $[L_*, \infty)$ and $L_* = L$, define Y_* to be the union of the components of the constant $s \in \mathbb{I}$ slices of X where $c_1(\det(\mathbb{S}))$ is not torsion. In the case when $L_* = L_{tor}$, define Y_* to be the union of the components of the constant $s \in \mathbb{I}$ slices of X where $c_1(\det(\mathbb{S}))$ is torsion. Write $\mathbb{A}$ on $\mathbb{I} \times Y_*$ as $A_K + 2A$ and introduce by way of notation $\mathfrak{d}|_s$ to denote the pull-back to $\{s\} \times Y_*$ of (A, ψ). Also introduce $\mathfrak{B}_{(A,\psi)}$ to denote

$$\mathfrak{B}_{(A,\psi)} = B_A - \mathrm{r}(\psi^\dagger\tau\psi - i * w) + i * d\mu_* + \frac{1}{2}B_{A_K}, \tag{8.14}$$

with μ_* denoting either μ_- or μ_+ as the case may be. Use D_A in what follows to denote the Dirac operator on Y_* as defined using the connection $A_K + 2A$ for the Spin^c-structure with spinor bundle $\mathbb{S} = \mathbb{S}^+$. Suppose that $s' > s$ are two points in $\mathbb{I}$. Take the L^2-norm of the left hand expressions in both equations of (8.13) over $[s, s'] \times Y_*$. The square of these norms are zero. This being the case, integration by parts in the expressions the square of these L^2-norms results in an identity of the form

$$\frac{1}{2}\int_{[s,s']\times Y_*} \big(|\frac{\partial}{\partial s}A_*|^2 + |\mathfrak{B}_{(A,\psi)}|^2 + 2\mathrm{r}\big(|\frac{\partial}{\partial s}\psi_*|^2 + |D_A\psi|^2\big)\big) = \mathfrak{a}(\mathfrak{d}|_s) - \mathfrak{a}(\mathfrak{d}|_{s'}). \tag{8.15}$$

Taking limits in (8.15) as $s \to -\infty$ or as $s' \to \infty$ as the case may be leads to the identities

(8.16)
$$\begin{cases} \frac{1}{2}\int_{\mathbb{I}\times Y_*}(|\frac{\partial}{\partial s}A_*|^2 + |\mathfrak{B}_{(A,\psi)}|^2 + 2\mathrm{r}(|\frac{\partial}{\partial s}\psi_*|^2 + |D_A\psi|^2)) = \mathfrak{a}(\mathfrak{c}_-) - \mathfrak{a}(\mathfrak{d}|_{s=-L_*}). \\ \frac{1}{2}\int_{\mathbb{I}\times Y_*}(|\frac{\partial}{\partial s}A_*|^2 + |\mathfrak{B}_{(A,\psi)}|^2 + 2\mathrm{r}(|\frac{\partial}{\partial s}\psi_*|^2 + |D_A\psi|^2)) = \mathfrak{a}(\mathfrak{d}|_{s=L_*}) - \mathfrak{a}(\mathfrak{c}_+). \end{cases}$$

Note that the identities in (8.15) and (8.16) hold with $\mathfrak{d} = (A_K + 2A, \psi)$ on the right hand side. By way of an explanation, the integration by parts proves the analogs that have $\mathfrak{d}_* = (A_K + 2A_*, \psi_*)$ used on the right hand side, and if they hold using $\mathfrak{d}_*$, then they hold using $\mathfrak{d}$ because the restriction of the map $\hat{u}$ to any slice $\{s\} \times Y_*$ in $\mathbb{I} \times Y_*$ is homotopic to the constant map to S^1.

Step 3: The assertion made by the first bullet of Lemma 8.1 follows directly from (8.16) when the data is such that X is the product cobordism. The proof in the general case and the proof of the second bullet of Lemma 8.1 use an integral version of the Weitzenböck formula for the operator $\mathcal{D}_{\mathbb{A}}^{-}\mathcal{D}_{\mathbb{A}}^{+}$. The details follow directly.

Integrate $|F_{\mathbb{A}}^{+} - \mathrm{r}(\psi^\dagger\tau\psi - \frac{i}{2}w_X) + i\mathfrak{w}_\mu^{+}|^2 + 2\mathrm{r}|\mathcal{D}_{\mathbb{A}}^{+}\psi|^2$ over $s^{-1}([-L-3, L+3])$ and denote the result by $\mathcal{I}$. Integrate this same expression over the respective $|s| \in [L, L_* - 4]$ and $[L_* - 5, L_* + 1]$ parts of X_{tor}. Denote these integrals as $\mathcal{I}_{tor0}$ and $\mathcal{I}_{tor1}$. In each case, let X_* denote the region of integration and let $\partial_- X_*$ and $\partial_+ X_*$ denote the two boundaries of the relevant region of integration with $\partial_- X_*$ at the smaller value of s and $\partial_+ X_*$ at the larger value. Use the Weitzenböck formula for $\mathcal{D}_{\mathbb{A}}^{-}\mathcal{D}_{\mathbb{A}}^{+}\psi$ from Step 1 with Stokes' theorem to rewrite the identities $\mathcal{I} = 0$, $\mathcal{I}_{tor0} = 0$ and $\mathcal{I}_{tor1} = 0$ respectively as

$$\frac{1}{2}\int_{X_*} (|F_{\mathbb{A}}|^2 + \mathrm{r}^2|\psi^\dagger\tau\psi - iw_X|^2 + 2\mathrm{r}|\nabla_{\mathbb{A}}\psi|^2) + \mathfrak{i}_* = \mathfrak{a}(\mathfrak{d}|_{p_- X_*}) - \mathfrak{a}(\mathfrak{d}|_{p_+ X_*}), \tag{8.17}$$

with $\mathfrak{i}_*$ in the case of $\mathcal{I}$ and $\mathcal{I}_{tor1}$ denoting a term with absolute value no greater than $c_0 c\mathrm{r}(\int_{s^{-1}([-L-3, L+3])}(|F_{\mathbb{A}}|^2)^{1/2} + c_0 r^{1+c_0/c}$. In the case of $\mathcal{I}_{tor0}$, the absolute value of $\mathfrak{i}_*$ is no greater than $c_0 cL_*\mathrm{r}$. This bound on $|\mathfrak{i}_*|$ in the case of $\mathcal{I}$ and $\mathcal{I}_{tor0}$ is a direct consequence of the bounds on the norms of the Riemannian curvature tensor and w_X, the size of L, the volume of the s-inverse image of intervals, and the bound $|\psi|^2 \le 2c$ from Step 1. In the case of $\mathcal{I}_{tor0}$, the bound for $|\mathfrak{i}_*|$ is a consequence of the fact that $dw_X = 0$ on the integration domain, this being the assumption made by Item c) of the fourth bullet of (3.13). By way of an explanation, $\mathfrak{i}_*$ in this case can be written as sum of three terms, these denoted by $\mathfrak{i}_{\mathfrak{g}}$, $\mathfrak{i}_\mu$ and $\mathfrak{i}_K$. The term that is denoted by $\mathfrak{i}_{\mathfrak{g}}$ gives the contribution of the scalar curvature term in the Weitzenböck formula for $\mathcal{D}_{\mathbb{A}}^{-}\mathcal{D}_{\mathbb{A}}^{+}$. As such, it is bounded by the integral of $c_0\mathrm{r}|\psi|^2$ over the $|s| \in [L, L_* - 4]$ part of X_{tor}. The bound $|\psi|^2 \le c_0 c$ leads to a bound on $|\mathfrak{i}_{\mathfrak{g}}|$ by $c_0 c\mathrm{r}L_*$.

The term that is denoted by $\mathfrak{i}_\mu$ comes by writing $|F_{\mathbb{A}}^{+} - \mathrm{r}(\psi^\dagger\tau\psi - \frac{i}{2}w_X) + i\mathfrak{w}_\mu^{+}|^2$ as the sum of $|F_{\mathbb{A}}^{+} - \mathrm{r}(\psi^\dagger\tau\psi - \frac{i}{2}w_X)|^2$ with terms that involve w_μ^{+}. One of these terms has the inner product between $F_{\mathbb{A}}^{+}$ and $\mathfrak{w}_\mu^{+}$. Stokes' theorem identifies the integral of the latter with the contributions to the boundary terms on the right hand side of (8.17) from the $\mathfrak{e}_\mu$ part of the functional $\mathfrak{a}$. The other $\mathfrak{w}_\mu^{+}$ terms are bounded by the integral

over X_* of $c_0\big(\mathrm{r}\Big||\psi|^2 - |\frac{i}{2}w_X|\Big||\mathfrak{w}_\mu^+| + |\mathfrak{w}_\mu^+|^2\big)$. This understood, the bounds on $|\psi|^2$ and $|w_X|$ lead to a bound on $|\mathfrak{i}_\mu|$ by $c_0 c\mathrm{r}L_*$.

What follows explains how the term $\mathfrak{i}_K$ in $\mathfrak{i}_*$ arises. The $dw_X = 0$ assumption is used to derive a suitable bound on $|i_K|$. As noted above, the derivation starts by writing $|F_{\mathbb{A}}^+ - \mathrm{r}(\psi^\dagger\tau\psi - \frac{i}{2}w_X) + iw_\mu^+|^2$ as $|F_{\mathbb{A}}^+ - \mathrm{r}(\psi^\dagger\tau\psi - \frac{i}{2}w_X)|^2$ plus terms that involve $\mathfrak{w}_\mu^+$. The norm $|F_{\mathbb{A}}^+ - \mathrm{r}(\psi^\dagger\tau\psi - w_X)|^2$ is then written as a sum of $|F_{\mathbb{A}}^+|^2$, $\mathrm{r}^2|\psi^\dagger\tau\psi - \frac{i}{2}w_X|^2$ and twice the inner product between $F_{\mathbb{A}}^+$ and $\mathrm{r}(\psi^\dagger\tau\psi - \frac{i}{2}w_X)$. The integral over X_* of the term with the inner product between $F_{\mathbb{A}}^+$ and $\mathrm{r}\psi^\dagger\tau\psi$ is cancelled by the contribution from the $F_{\mathbb{A}}^+$ term in the Weitzenböck formula for $\mathcal{D}_{\mathbb{A}}^-\mathcal{D}_{\mathbb{A}}^+\psi$. The inner product between $F_{\mathbb{A}}^+$ and $-\frac{i}{2}\mathrm{r}w_X$ is equal to that of $F_{\mathbb{A}}$ with $-\frac{i}{2}\mathrm{r}w_X$ and thus its integral is that of $\mathrm{r}F_{\mathbb{A}} \wedge w_X$. Stokes' theorem identifies most of the latter with the contributions to the boundary terms on the right hand side of (8.17) from the rw term in $\mathfrak{a}$. The term designated by $\mathfrak{i}_K$ is what remains after the application of Stokes' theorem. To say more about $\mathfrak{i}_K$, note that the application here of Stokes' theorem requires writing $\mathbb{A}$ as $A_K + 2A_E + \hat{\mathrm{a}}_{\mathbb{A}}$ with $\hat{\mathrm{a}}_{\mathbb{A}}$ being an i-valued 1-form on X_*. Stokes' theorem involves only $\hat{\mathrm{a}}_A$. The $\mathfrak{i}_K$ term is the integral of $\frac{i}{2}\mathrm{r}F_{A_K+2A_E} \wedge w_X$. This understood, the bound $|\mathfrak{i}_K| \le c_0 c\mathrm{r}L_*$ follows from the $|w_X| \le c$ assumption.

There is one other subtle point with regards to the derivation of (8.17) in the case when X_* is the $|s| \le L+3$ part of X, this being that the application of Stokes' theorem requires a Hermitian connection on the bundle $\det(\mathbb{S}^+)$ whose curvature has norm bounded by $\mathrm{r}^{c_1/c}$ with c_1 being a constant that is independent of $\mathfrak{d}$, r, c, the metric and w_X. The pull back of this connection from the $s \le -L$ and $s \ge L$ part of X via the embeddings from the second and third bullets should also be the respective Y_- and Y_+ versions of $A_K + 2A_E$. Such a connection can be constructed using the isomorphism between de-Rham cohomology and the Čech cohomology that is defined by a cover of the $|s| \le L+1$ part of X by Gaussian coordinate charts with the property that the any given number of charts have either empty or convex intersection (see Chapter 8 in [BT]). The $\mathrm{r}^{1/c}$-bound on the norm of Riemannian curvature and the $\mathrm{r}^{-1/c}$ lower bound on the injectivity radius can be used to obtain such a cover by sets of radius greater than $\mathrm{r}^{-c_0/c}$. As the connection is constructed from the de-Rham isomorphism using a subbordinate partition of unity, this lower bound on the minimum chart radius can be used to construct a connection on $\det(\mathbb{S}^+)$ with an $\mathrm{r}^{c_0/c}$ bound on the norm of its curvature.

Section 8.6 says more about $\mathfrak{i}_*$ when the $(c, r = \mathrm{r})$ version of (3.13) is assumed.

Step 4: Define X_*, $\partial_- X_*$, and $\partial_+ X_*$ as in Step 3. Granted Step 3's bound for the norm of the $\mathfrak{i}_*$ term in (8.17), then (8.15) and (8.17) imply that

$$\mathfrak{a}(\mathfrak{d}|_{\partial_+ X_*}) \le \mathfrak{a}(\mathfrak{d}|_{\partial_- X_*}) + c_0 c^2 \mathrm{r}^2. \tag{8.18}$$

This inequality with the top identity in (8.16) imply that $\mathfrak{a}(\mathfrak{c}_+) \le \mathfrak{a}(\mathfrak{d}|_s) \le \mathfrak{a}(\mathfrak{c}_-) + c_0 c^2 \mathrm{r}^2$ when $s \ge L$; and the identity in the bottom bullet of (8.16) and (8.18) imply the inequalities $\mathfrak{a}(\mathfrak{c}_-) \ge \mathfrak{a}(\mathfrak{d}|_s) \ge \mathfrak{a}(\mathfrak{c}_+) - c_0 c^2 \mathrm{r}^2$ when $s \le -L$. Given these inequalities, then (8.17) implies that

$$\frac{1}{2}\int_{X_*} (|F_{\mathbb{A}}|^2 + \mathrm{r}^2|\psi^\dagger \tau \psi - i w_X|^2 + 2\mathrm{r}|\nabla_{\mathbb{A}}\psi|^2) \le \mathfrak{a}(\mathfrak{c}_-) - \mathfrak{a}(\mathfrak{c}_+) + c_0 c^2 \mathrm{r}^2. \tag{8.19}$$

This last inequality with the identities in (8.15) and (8.16) imply directly the assertion made by the first bullet of Lemma 8.1 and it implies the second bullet when the length one interval is part of $[-L-3, L+3]$ or $[-L_* - 1, -L_* + 5]$ or $[L_* - 5, L_* + 1]$.

Granted what was just said, the second bullet of Lemma 8.1 holds if its assertion is true when the length one interval is disjoint from $[-L, L]$, $[-L_*, -L_* + 4]$ and $[L_* - 4, L_*]$. To prove the assertion for these cases, use (8.18) with (8.15) and (8.16) to see that $\mathfrak{a}(\mathfrak{d}|_s) - \mathfrak{a}(\mathfrak{d}|_{s'}) < c_0 c^2 \mathrm{r}^2$ if $s > s'$ and if both are in the same component of the complement in $\mathbb{R}$ of any of these three intervals. This fact is exploited for the case $s' = s + 1$ using an integration by parts argument to rewrite the integrand on the left hand side of the $s' = s + 1$ version of (8.15) so as to have the same form as the integrand on the left hand side of (8.17). The resulting inequality with the bound $\mathfrak{a}(\mathfrak{d}|_s) - \mathfrak{a}(\mathfrak{d}|_{s+1}) < c_0 c^2 \mathrm{r}^2$ leads directly to what is asserted by Lemma 8.1's second bullet. □

8.4 Proof of Lemma 8.2

The proof of Lemma 8.2 has five steps. By way of a look ahead, the arguments depend crucially on the fact that the metric with the 2-form $ds \wedge *w + w$ define a Kähler structure on U_C and on U_0. The proof that follows considers only the special case where both μ_- and μ_+ vanish on the respective Y_- and Y_+ versions of U_γ and $\mathcal{H}_0$. The argument in the general case is little different and so not given.

Step 1: Let V_* denote either U_C or U_0. The fact that the metric with $ds \wedge *w + w$ defines an integrable complex structure on V_* has following consequence: View β as

a section of the $(0,2)$-part of $\wedge^2 T^* V_* \otimes \mathbb{C}$. Then the right most equation in (2.9) can be written on either U_C or U_0 as

$$\bar{\partial}_A \alpha + \bar{\partial}_A^{\dagger} \beta = 0. \tag{8.20}$$

This last equation implies that β obeys

$$\nabla_A^{\dagger} \nabla_A \beta + \mathrm{r}(1 + |\alpha|^2 + |\beta|^2)\beta + \mathfrak{r}\beta = 0, \tag{8.21}$$

where $\mathfrak{r}$ is determined solely by the metric. In particular, the absolute value of $\mathfrak{r}$ and its derivatives to any specified order are also bounded by c_0. The equation just written implies that $|\beta|^2$ obeys the differential inequality

$$d^{\dagger} d|\beta|^2 + \mathrm{r}|\beta|^2 + |\nabla_A \beta|^2 \le 0. \tag{8.22}$$

This last inequality is exploited momentarily with the help of the Green's function for the operator $d^{\dagger} d + \mathrm{r}$.

Let $x \in V_*$ denote a given point and let $G_x(\cdot)$ denote the Dirichlet Green's function for $d^{\dagger} d + \mathrm{r}$ with pole at x. Keep in mind for what follows the following fact about $G_x(\cdot)$: It is non-negative and it obeys:

$$G_x(\cdot) \le c_0 \frac{1}{\mathrm{dist}(x,\cdot)^2} e^{-\sqrt{\mathrm{r}}\,\mathrm{dist}(x,\cdot)}. \tag{8.23}$$

Introduce $\mathrm{D}\colon V_* \to [0, c_0)$ to denote the function that measure the distance to the boundary of V_*. Fix x in the interior of D_*, multiply both sides of (8.22) by $G_x(\cdot)$ and integrate the resulting inequality over V_*. An integration by parts in the left hand integral using the bound $|\beta|^2 \le c_0 c$ from Lemma 8.1 leads directly to the following inequalities:

$$\begin{cases} |\beta|^2 \le c_0 c e^{-\sqrt{\mathrm{r}}\,\mathrm{D}}, \\ \int_B G_x |\nabla_A \beta|^2 \le c_0 c \frac{1}{\mathrm{D}^2} e^{-\sqrt{\mathrm{r}}\,\mathrm{D}}. \end{cases} \tag{8.24}$$

The second inequality is used in Step 3 to derive bounds on the higher order derivatives of β.

Step 2: This step constitutes a digression to state some very crude bounds for the norms of $F_{\mathbb{A}}$, $\nabla_{\mathbb{A}} \psi$ and their covariant derivatives. The following lemma states these bounds.

Lemma 8.4 *There exists $\kappa > \pi$ such that given any $c > \kappa$, there exists κ_c with the following significance: Fix $\mathrm{r} \ge \kappa_c$ and assume the $(c, r = \mathrm{r})$ version of the first two*

bullets of (3.13). Assume in addition that $|w_X| \le c$ and that the norms of its derivatives to order 10 are bounded by $\mathrm{r}^{1/c}$. Fix respective elements μ_- and μ_+ from the Y_- and Y_+ versions of Ω with $\mathcal{P}$-norm bounded by 1. Use this data to define the equations in (2.9). Let $\mathfrak{d} = (\mathbb{A}, \psi)$ denote an instanton solution to (2.9) with $F_{\mathbb{A}}$ and $\mathrm{r}^{1/2}|\nabla_{\mathbb{A}}\psi|$ having L^2-norm less than $c\mathrm{r}$ on the s-inverse image of any length 1 interval in $\mathbb{R}$. Then the norm of $F_{\mathbb{A}}$ and $|\nabla_{\mathbb{A}}\psi|$, and those of their derivatives up through order 4 are bounded everywhere by $\kappa_c \mathrm{r}^{\kappa_c}$.

Proof. This follows using a standard elliptic boot-strapping argument since the equations in (2.9) can be viewed as elliptic equations on any given ball in X for a suitable pair on the $C^\infty(X; S^1)$-orbit of $(\mathbb{A}, \psi)$. Except for one remark, the details of this bootstrapping are completely straightforward and so they will not be presented. The remark concerns the fact that the assumed lower bound for the injectivity radius is needed for the proof so as to invoke various Sobolev embedding theorems using embedding constants that are bounded by powers of r. □

The bounds supplied by Lemma 8.4 are used in the next step.

Step 3: To obtain the asserted bound for the covariant derivative of β, differentiate (8.21) and commute covariant derivatives to obtain an equation for $\nabla_A\beta$ that has the schematic form

$$\begin{aligned}\nabla_A^\dagger\nabla_A(\nabla_A\beta) + \mathrm{r}(1 + |\alpha|^2 + |\beta|^2)\nabla_A\beta +& \\ \mathfrak{R}_0(F_A)\nabla_A\beta + \mathfrak{R}_1(\nabla F_A)\beta + \mathrm{r}\mathfrak{R}_2(\nabla_A\psi)\nabla_A\beta + \mathfrak{r}_1\nabla_A\beta &= 0,\end{aligned} \tag{8.25}$$

where $\mathfrak{R}_0$, $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are endomorphisms that are linear functions of their entries and are such that $|\mathfrak{R}_*(b)| \le c_0|b|$. Meanwhile, $\mathfrak{r}_1$ is such that $|\mathfrak{r}_1| \le c_0$ also. Take the inner product of both sides of (8.25) with $\nabla_A\beta$ and invoke Lemma 8.4 to see that

$$d^\dagger d(|\nabla_A\beta|^2) + \mathrm{r}|\nabla_A\beta|^2 + |\nabla_A\nabla_A\beta|^2 \le c_c\mathrm{r}^{c_c}(|\nabla_A\beta|^2 + |\beta|^2), \tag{8.26}$$

where c_c here and in what follows denotes a constant that is greater than 1 and depends only on c. The value of c_c can be assumed to increase between consecutive appearances.

Fix a point $x \in V_*$ with distance greater than $c_0\mathrm{r}^{-1/2}(\ln \mathrm{r})^2$ from the boundary of V_*. Having done so, multiply both sides of (8.26) by G_x and integrate both sides of V_*. Use the second bullet in (8.24) to bound integral on the right hand side of the resulting inequality by $c_0 e^{-\sqrt{\mathrm{r}}/c_0}$ when $\mathrm{r} \ge c_c$. An integration by parts on the left hand side

using Lemma 8.4 to bound $|\nabla_A\beta|$ on the boundary of V_* and the bound just stated implies that

$$|\nabla\beta_A|^2(x) + \int_B G_x|\nabla_A\nabla_A\beta|^2 \leq c_0 e^{-\sqrt{\mathrm{r}}/c_0} \tag{8.27}$$

when $\mathrm{r} \geq c_c$. This gives the desired bound for $|\nabla_A\beta|$.

To obtain the bound for $|\nabla_A\nabla_A\beta|$, differentiate (8.25) twice and take the inner product of both sides with $\nabla_A\nabla_A\beta$ after commuting covariant derivatives. The result is an equation that looks much like (8.26) with $\nabla_A\beta$ replaced by $\nabla_A\nabla_A\beta$ on the left hand side and with the addition of the term $\mathrm{r}^{c_c}|\nabla_A\nabla_A\beta|^2$ on the right hand side. Granted that this is the case, then the same Green's function argument that led to (8.27) leads to an analogous bound for $|\nabla_A\nabla_A\beta|^2$.

Step 4: This step and Step 5 addresses the assertions of Lemma 8.2 that concern α. To start, act by $\overline{\partial}_A^\dagger$ on both sides of (8.20), commute covariant derivatives and use the bounds from Lemma 8.2 for $|\beta|$ to see that α obeys an equation that has the form

$$\nabla_A^\dagger\nabla_A\alpha - \mathrm{r}(1-|\alpha|^2)\alpha = \mathfrak{e}, \tag{8.28}$$

where $|\mathfrak{e}| \leq e^{-\sqrt{\mathrm{r}}/c_0}$ when $\mathrm{r} \geq c_c$. This equation implies that $\mathrm{w} = 1 - |\alpha|^2$ obeys a differential inequality of the form

$$d^\dagger d\mathrm{w} + \mathrm{r}\mathrm{w} \geq |\nabla_A\alpha|^2 + \mathrm{r}\mathrm{w}^2 - e^{-\sqrt{\mathrm{r}}/c_0}. \tag{8.29}$$

Use of the Green's function G_x with the fact that $|\mathrm{w}| \leq c_0 c$ on the boundary of V_* along the same lines as in Parts 1 and 3 finds $\mathrm{w} \geq e^{-\sqrt{\mathrm{r}}/c_0}$ at distances greater than $c_0\mathrm{r}^{-1/2}(\ln\mathrm{r})^2$ from the boundary of V_* when $\mathrm{r} \geq c_c$. This is the $|\alpha|^2$ assertion in the first bullet of Lemma 8.2.

The assertion in the third bullet follows directly from (8.20) given Lemma 8.2's bounds for $|\beta|$ and $|\nabla_A\beta|$. The assertion in the fourth bullet follows directly from (8.29) given that $\mathrm{w}(1-\mathrm{w}) = |\alpha|^2(1-|\alpha|^2)$ and that this is greater than $\frac{1}{2}\delta^2$ at points where $|\alpha|^2$ is between δ and $1-\delta$. The assertions in the fifth bullet about the covariant derivatives of α are proved in Step 5.

Step 5: This step derives the asserted bounds in the fifth bullet for the norms of the covariant derivatives of α. To do this, suppose that $x \in V_*$ is such that $|F_A| \leq c\mathrm{r}$ on the ball of radius $c_0\mathrm{r}^{-1/2}$ centered at x. Use r_r in what follows to denote the rescaling map from $\mathbb{C}^2$ to $\mathbb{C}^2$ that is given by the rule $x \mapsto r_\mathrm{r}(x) = \mathrm{r}^{-1/2}x$. The pull-back of (A,ψ) by this map is denoted by $(A_\mathrm{r},\psi_\mathrm{r})$. The bound $|F_A| \leq c\mathrm{r}$ implies that the absolute value

of the curvature of A_{r} is bounded in the radius 1 ball about the origin in $\mathbb{C}^2$ is bounded by c. Meanwhile, the pull-back of the equations in (2.9) by this map constitutes a uniformly elliptic system of equations (modulo the action of $C^\infty(\mathbb{C}^2; S^1)$ in the radius 1 ball about the origin in $\mathbb{C}^2$ with coefficients that have r-independent bounds for their absolute values and for those of their derivatives to any a priori chosen order. This understood, the fact that $|\psi_{\mathrm{r}}| \leq 2$ in this ball and the afore-mentioned bound by c for the norm of the curvature of A_{r} imply via standard elliptic bootstrapping arguments that the A_{r}-covariant derivatives of ψ_{r} through order 2 are bounded by $c_0 c$ in the radius c_0^{-1} ball about the origin in $\mathbb{C}^2$. Granted these bounds, use the chain rule of calculus to obtain the bounds asserted by the fifth bullet of Lemma 8.2 for the covariant derivative of α. □

8.5 Proof of Lemma 8.3

Use V_* again to denote either U_C or U_0. The functions $\mathfrak{z}_A$ and $\mathfrak{z}_B$ are both equal to $\mathrm{r}|\beta|^2$ on V_* and so what is asserted by the second bullet of Lemma 8.3 follows from the first bullet of Lemma 8.2. The absolute value of r is bounded by $c_0\mathrm{r}|\alpha||\beta|$ on V_* and so the third bullet of Lemma 8.3 also follows from the first bullet of Lemma 8.2. The bounds in the first bullet of Lemma 8.3 follow from the bound in the fourth bullet and that for $|\alpha|^2$ in the first bullet of Lemma 8.2. If the bounds in first through fourth bullets of Lemma 8.3 hold, then $|F_A|$ is bounded by $c_0\mathrm{r}$ at the points in V_* with distance $\frac{1}{200}\rho_D$ from the boundary of V_*. Granted that this is the case, then the rescaling argument in Step 5 of the proof of Lemma 8.2 can be used to derive the bound given in the fifth bullet of Lemma 8.3.

The upcoming Lemma 8.5 is the critical ingredient for the proof of the fourth bullet of Lemma 8.3. The $\mathfrak{a}(\mathfrak{c}_-) - \mathfrak{a}(\mathfrak{c}_+) \leq \mathrm{r}^{2-1/c}$ assumption in Lemma 8.3 and the final three bullets of (3.13) are needed only to invoke Lemma 8.5.

Lemma 8.5 *There exists $\kappa > 100(1+\rho_D^{-1})$ such that given any $c \geq \kappa$, there exists $\kappa_c > \kappa$ with the following significance: Fix $\mathrm{r} \geq \kappa_c$ and assume that the metric and w_X are $(c, r = \mathrm{r})$-compatible. Fix elements μ_- and μ_+ from the Y_- and Y_+ versions of Ω with $\mathcal{P}$-norm bounded by 1 and use this data to define the equations in (2.9). Let $\mathfrak{c}_-$ and $\mathfrak{c}_+$ denote solutions to the (r, μ_-)-version of (2.4) on Y_- and the (r, μ_+) version of (2.4) on Y_+ with $\mathfrak{a}(\mathfrak{c}_-) - \mathfrak{a}(\mathfrak{c}_+) \leq \mathrm{r}^{2-1/c}$. Let $\mathfrak{d} = (\mathbb{A}, \psi)$ denote an instanton solution to (2.9) with $s \to -\infty$ limit equal to $\mathfrak{c}_-$ and $s \to \infty$ limit equal to $\mathfrak{c}_+$. Use B to denote a ball of radius κ^{-2} in the domain U_C or in the domain U_0 with center at distance κ^{-1} or more from the domain's boundary. Then $\mathrm{r}\int_B |1-|\psi|^2| \leq \kappa_c \mathrm{r}^{1-1/\kappa_c}$.*

Lemma 8.5 is proved in Section 8.6. Granted Lemma 8.5, then the six steps that follow prove the fourth bullet of Lemma 8.3 in the case when μ_- and μ_+ are zero on the Y_- and Y_+ version of U_γ and $\mathcal{H}_0$. The proof when they are not zero but bounded by $e^{-\mathrm{r}^2}$ is little different and so not given.

Step 1: Let V_* denote either U_C or U_0. Keep in mind that metric on V_* has non-negative Ricci curvature tensor, that the 2-form $w_X = w$ is covariantly constant on V_*, that both $\mathfrak{w}_\mu = 0$ and that B_{A_K} is covariantly constant on V_*. These facts with the bounds from Lemma 8.2 for $|\beta|$ and $|\nabla_A\beta|$ have the following implication: Let $\mathfrak{s}$ denote $|\mathcal{E}_A - \mathcal{B}_A|$. Granted that $\mathrm{r} \geq c_c$, then the equations in (2.9) imply that $\mathfrak{s}$ obeys the differential inequality

$$d^\dagger d\mathfrak{s} + \mathrm{r}|\alpha|^2\mathfrak{s} \leq \mathrm{r}|\nabla_A\alpha|^2 + e^{-\sqrt{\mathrm{r}}/c_0} \tag{8.30}$$

at the points in V_* with distance greater than $c_0\mathrm{r}^{-1/2}(\ln \mathrm{r})^2$ from the boundary of V_*. Let w again denote $1 - |\alpha|^2$ and let q_0 denote $\mathfrak{s} - \mathrm{rw}$. It follows from (8.29) and (8.30) that

$$d^\dagger dq_0 + \mathrm{r}|\alpha|^2 q_0 \leq e^{-\sqrt{\mathrm{r}}/c_0} \tag{8.31}$$

at the points in V_* with distance $c_0\mathrm{r}^{-1/2}(\ln \mathrm{r})^2$ or more from V_*'s boundary if $\mathrm{r} \geq c_c$.

Step 2: Fix $\rho_* > 0$ but such that $\rho_* < 10^{-8}\rho_D$. Fix $s_0 \in \mathbb{R}$. Let $V' \subset V_*$ denote the set of points in the $(s_0 - 1 - \rho_*, s_0 + 1 + \rho_*)$ part of V_* with distance ρ_* or more from the boundary of V_*, and let $V \subset V'$ denote the set of points in V_* with distance greater than $2\rho_*$ from the boundary of V_*. Thus, each point in V has distance ρ_* or more from the boundary of V'.

Fix a sequence $\{\varsigma_n\}_{n=1,\ldots}$ of smooth, non-negative functions on V' with the following properties: Each function in this series is bounded by 1 and is equal to 1 on V. Second, ς_1 has compact support and for each $n \geq 1$, the function ς_{n+1} has compact support where $\varsigma_n = 1$. Finally, the absolute values of the first and second derivatives of the functions in this series enjoy s_0-independent upper bounds.

Step 3: For each integer $n \geq 1$, set $q_n = \max(\varsigma_n q_{n-1}, 0)$. Use $\mathfrak{q}_0$ to denote the maximum of q_0; and for $n \geq 1$, use $\mathfrak{q}_n$ to denote the maximum of q_n. Note that $\mathfrak{q}_n \leq \mathfrak{q}_{n-1}$. It follows from (8.31) that if $\mathrm{r} \geq c_c$, then any given $n \geq 1$ version of q_n obeys

$$d^\dagger dq_n + \mathrm{r}|\alpha|^2 q_n \leq (dd^\dagger\varsigma_n)\, q_{n-1} + 2\langle d\varsigma_n, dq_{n-1}\rangle + c_0 e^{-\sqrt{\mathrm{r}}/c_0}, \tag{8.32}$$

where $\langle\cdot,\cdot\rangle$ denotes the metric inner product. Fix a constant $z_n \geq 1$ to be determined shortly, and let q_{n*} denote the maximum of 0 and $q_n - \mathrm{r}^{-1} z_n q_{n-1}$. The function q_{n*} obeys

$$
\begin{aligned}
&d^\dagger d q_{n*} + \mathrm{r}|\alpha|^2 q_{n*} \\
&\quad \leq z_n q_{n-1} \mathrm{w} + \big(- z_n q_{n-1} + (dd^\dagger \varsigma_n)\, q_{n-1} + 2\langle d\varsigma_n, dq_{n-1}\rangle\big) + c_0 e^{-\sqrt{\mathrm{r}}/c_0}.
\end{aligned} \tag{8.33}
$$

Note also that q_{n*} has compact support in V' since $q_n - \mathrm{r}^{-1} z_n q_{n-1} = -\mathrm{r}^{-1} z_n q_{n-1}$ on the complement of the support of ς_n .

Step 4: Fix x in the interior of V' and let G_x now denote the Dirichlet Green's function for the operator $d^\dagger d$ on V' with pole at x. The function G_x is non-negative, $|G_x(\cdot)| \leq c_0 \,\mathrm{dist}(x,\cdot)^{-2}$ and $|dG_x(\cdot)| \leq c_0 \,\mathrm{dist}(x,\cdot)^{-3}$. Multiply both sides of (8.33) by G_x and integrate the two sides of the resulting inequality over V'. Integrate by parts on both sides to remove derivatives from q_{n*} and q_{n-1} to obtain the inequality

$$
q_{n*}(x) \leq z_n q_{n-1} \int_{V'} \Big(\frac{1}{\mathrm{dist}(x,\cdot)^2} \mathrm{w}\Big) + (-c_0^{-1} z_n + e_n) q_{n-1} + e^{-\sqrt{\mathrm{r}}/c_0}. \tag{8.34}
$$

where $e_n \leq c_0 \sup_{x\in V'}(|d^\dagger d\varsigma_n| + |d\varsigma_n|)$. Granted this bound, a purely n-dependent choice for z_n leads from (8.34) to the inequality

$$
q_{n*}(x) \leq z_n q_{n-1} \int_{V'} \Big(\frac{1}{\mathrm{dist}(x,\cdot)^2} \mathrm{w}\Big) + e^{-\sqrt{\mathrm{r}}/c_0}, \tag{8.35}
$$

Lemma 8.5 is used to exploit this inequality.

Step 5: Fix $\rho > 0$ and break up the integral in (8.35) into the part where $\mathrm{dist}(x,\cdot)$ is greater than ρ and the part where $\mathrm{dist}(x,\cdot)$ is less than ρ. Having done so, appeal to Lemma 8.5 and the first bullet of Lemma 8.2 to see that

$$
q_{n*}(x) \leq z_n(\rho^{-2} \mathrm{r}^{-1/c_0} + \rho^2)\, q_{n-1} + e^{-\sqrt{\mathrm{r}}/c_0} \tag{8.36}
$$

when $\mathrm{r} \geq c_c$. Let c_* denote the value of c_0 that appears in (8.36). Take $\rho = \mathrm{r}^{-1/4c_*}$ in (8.36). The resulting right hand side is independent of x; and this leads directly to the inequality

$$
q_n \leq z_n \mathrm{r}^{-1/2c_*} q_{n-1} + e^{-\sqrt{\mathrm{r}}/c_0} \tag{8.37}
$$

when $\mathrm{r} \geq c_c$. As Lemma 8.4 finds $q_0 < \mathrm{r}^{c_c}$, what is written in (8.37) implies that an $n = c_c$ version of q_n is bounded by r^{-200}.

Step 6: Since $\varsigma_n = 1$ on V, the conclusion from Step 5 implies that

$$|\mathcal{E}_A - \mathcal{B}_A| < \mathrm{r}(1 - |\alpha|^2) + \mathrm{r}^{-200} \tag{8.38}$$

at all points in V. Square both sides of (8.38). What with the bounds for $|\mathfrak{z}_A|$ and $|\mathfrak{z}_B|$ from Lemma 8.3's second bullet, the resulting inequality implies that

$$(1 - 2\sigma)^2\mathrm{r}^2(1 - |\alpha|^2) + |\mathfrak{X}|^2 \leq \mathrm{r}^2(1 - |\alpha|^2) + c_0\mathrm{r}^{-198}; \tag{8.39}$$

and rearranging terms writes this as

$$|\mathfrak{X}|^2 \leq 2\mathrm{r}^2\sigma(1 - \sigma)(1 - |\alpha|^2) + c_0\mathrm{r}^{-198}. \tag{8.40}$$

This is gives the bound stated in the fourth bullet of Lemma 8.3. □

8.6 Proof of Lemma 8.5

The proof has six parts. Parts 1 and 2 revisit the formula in (8.15) and Part 3 revisits the formula in (8.17). These steps present the proof in the case when $c_1(\det(\mathbb{S}))$ is non-torsion on all components of the $|s| > 1$ part of X. But for the two remarks that follow, the proof when $X_{tor} \neq \emptyset$ differs only cosmetically.

The first remark concerns the formula in (8.17) in the case when X_* is the respective $|s| \in [L, L_* - 4]$ part of X_{tor}, the remark being that the absolute value of $\mathfrak{i}_*$ in this case is bounded by $c_0 c^2 \mathrm{r} \ln \mathrm{r}$. The reason is as follows: As noted subsequent to (8.17), the absolute value of the relevant version of $\mathfrak{i}_*$ is bounded in any event by $c_0 c \mathrm{r} L_*$. Meanwhile, the first bullet of (3.13) bounds L_* by $c \ln \mathrm{r}$.

The second remark concerns (8.17) in the case when X_* is the $|s| \in [L_* - 4, L_*]$ part of X_{tor}, this being that the absolute value of the corresponding version of $\mathfrak{i}_*$ is at most c_0 when r is larger than a purely c-dependent constant. Given Item d) of the fourth bullet of (3.13), the proof that this is so differs only in notation from what is said below in Part 2 to prove the analogous bound for the version of $\mathfrak{i}_*$ that appears in (8.17) when X_* is the $|s| \in [L - 4, L]$ part of X.

Part 1: Write $\mathfrak{d} = (\mathbb{A}, \psi)$. When X, the metric and w_X are described by the first bullet of (3.14), use this pair as instructed in the proof of Lemma 8.1 to define the map (A_*, ψ_*) from $\mathbb{R}$ to $\mathrm{Conn}(E) \times C^\infty(Y_Z; \mathbb{S})$. When the second bullet of (3.14) is relevant, then (A_*, ψ_*) as defined in the proof of Lemma 8.1 denotes a map from $(-\infty, -1]$ to $\mathrm{Conn}(E|_{Y_-}) \times C^\infty(Y_-; \mathbb{S}|_{Y_-})$ and also a map from $[1, \infty)$ to $\mathrm{Conn}(E|_{Y_+}) \times C^\infty(Y_+; \mathbb{S}|_{Y_+})$.

Set $I_L = [-L, L]$ when X, the metric and w_X are described by the first bullet of (3.14), and set I_L to be either $[-L, -L+4]$ or $[L-4, L]$ otherwise. Use Y_* to denote the constant $s \in I_L$ slice of X, this being either Y_Z, Y_- or Y_+. Write the metric on $I_L \times Y_*$ as $ds^2 + \mathfrak{g}$ with $\mathfrak{g}$ denoting an s-dependent metric on Y_*. Define the s-dependent 1-form w_* on Y_* by writing w_X as $ds \wedge *w_* + w_*$ with the Hodge dual defined here by $\mathfrak{g}$. The two equations in (2.9) on the $s \in I_L$ part of X are equivalent to equations for (A_*, ψ_*) that can be written as

$$\text{(8.41)} \qquad \begin{cases} \frac{\partial}{\partial s} A_* + \mathfrak{B}_{\mathfrak{d}} = 0 & \text{and} \\ \frac{\partial}{\partial s}\psi_* + D_{A_*}\psi_* = 0, \end{cases}$$

with $\mathfrak{B}_{\mathfrak{d}}$ denoting the following $s \in I_L$ dependent 1-form on Y_*:

$$\text{(8.42)} \qquad \mathfrak{B}_{\mathfrak{d}} = B_{\mathbb{A}} - \mathrm{r}(\psi^\dagger \tau \psi - iw_*) + i\mathfrak{w}_\mu^+(\frac{\partial}{\partial s}, \cdot) + \frac{1}{2} B_{A_K}.$$

By way of notation, D_{A_*} in (8.41) denotes the Dirac operator defined by the metric $\mathfrak{g}$, its Levi-Civita connection and the connection $A_K + 2A_*$ on the $\{s\} \times Y_*$ version of $\det(\mathbb{S})$.

Part 2: If X, the metric and w_X are described by the first bullet of (3.14), then the integration and use of Stokes' theorem that leads to (8.15) can be repeated with the domain of integration being $s^{-1}([-L, L])$ to find that

$$\text{(8.43)} \qquad \begin{aligned} \frac{1}{2}\int_{\mathbb{R}\times Y_Z} \big(|\frac{\partial}{\partial s} A_*|^2 + |\mathfrak{B}_{(A,\psi)}|^2 + 2\mathrm{r}(|\frac{\partial}{\partial s}\psi_*|^2 + |D_A\psi|^2)\big) + \mathfrak{i}_\mu \\ = \mathfrak{a}(\mathfrak{d}|_{s=-L}) - \mathfrak{a}(\mathfrak{d}|_{s=L}), \end{aligned}$$

where $\mathfrak{i}_\mu = 0$ when $\mathfrak{w}_\mu$ is such that X, the metric, w_X and $\mathfrak{w}_\mu$ define the product metric, and where $|\mathfrak{i}_\mu| \le c_0\big(\int_{s^{-1}([-L,L])} |\frac{\partial}{\partial s} A_*|^2\big)^{1/2}$ in any event. This being the case, the second bullet of Lemma 8.1 implies that $|\mathfrak{i}_\mu| \le c_0\mathrm{r}$.

Assume now that X, the metric and w_X are described by the second bullet in (3.14). The derivation of (8.15) and (8.43) can be repeated with the domain of integration being $s^{-1}([-L, -L+4])$ and also $s^{-1}([L-4, L])$ to obtain the following identities:

$$\text{(8.44)} \qquad \begin{cases} \frac{1}{2}\int_{[-L,-L+4]\times Y_Z} \big(|\frac{\partial}{\partial s} A_*|^2 + |\mathfrak{B}_{(A,\psi)}|^2 + 2\mathrm{r}(|\frac{\partial}{\partial s}\psi_*|^2 + |D_A\psi|^2)\big) + \mathfrak{i} \\ \qquad = \mathfrak{a}(\mathfrak{d}|_{s=-L}) - \mathfrak{a}(\mathfrak{d}|_{s=-L+4}), \\ \frac{1}{2}\int_{[L-4,L]\times Y_Z} \big(|\frac{\partial}{\partial s} A_*|^2 + |\mathfrak{B}_{(A,\psi)}|^2 + 2\mathrm{r}(|\frac{\partial}{\partial s}\psi_*|^2 + |D_A\psi|^2)\big) + \mathfrak{i} \\ \qquad = \mathfrak{a}(\mathfrak{d}|_{s=L-4}) - \mathfrak{a}(\mathfrak{d}|_{s=L}), \end{cases}$$

where $\mathfrak{i}$ in this case is such that $|\mathfrak{i}| \le c_0 \mathrm{r}^{2-1/c}$ when $c > c_0$ and $\mathrm{r} > c_c$ with c_c denoting a constant that depends only on c. The paragraphs that follow explain how this bound comes about.

The term denoted by $\mathfrak{i}$ can be written as the sum of three integrals, $\mathfrak{i} = \mathfrak{i}_{\mathfrak{g}} + \mathfrak{i}_w + \mathfrak{i}_\mu$. What is denoted by $\mathfrak{i}_\mu$ appears here for the same reason it appears in (8.43) and it has the analgous bound, $|\mathfrak{i}_\mu| \le c_0 \mathrm{r}$. The integral denoted by $\mathfrak{i}_{\mathfrak{g}}$ accounts for the s-dependence of the metric $\mathfrak{g}$ on Y_* when commuting the operators $\frac{\partial}{\partial s}$ and D_{A_*}. In particular, the integrand that defines $\mathfrak{i}_{\mathfrak{g}}$ is bounded by $c_0 \mathrm{r}\big(|\frac{\partial}{\partial s}\mathfrak{g}||\psi||\nabla_A \psi| + |\mathfrak{R}_{\mathfrak{g}}(\frac{\partial}{\partial s}, \cdot)|, |\psi|^2\big)$ with $\mathfrak{R}_g$ denoting the Riemannian curvature tensor of the metric $ds^2 + \mathfrak{g}$. This understood, (3.13) with Lemma 8.1's bounds for $|\psi|^2$ and the L^2-norm of $|\nabla_A \psi|$ imply that $|\mathfrak{i}_{\mathfrak{g}}| \le c_0 \mathrm{r}^{3/2+1/c}$.

The integral $\mathfrak{i}_w$ arises from the contribution to the integral of $|\frac{\partial}{\partial s} A_* + \mathfrak{B}_{\mathfrak{d}}|^2$ of the metric inner product of $\frac{\partial}{\partial s} A_*$ with $-i\mathrm{r} * w_*$. The integral of this inner product is written as $\int_{I_L} h(s) ds$ with I_L denoting $[-L, -L+4]$ or $[L-4, L]$ as the case may be; and with $h(s)$ denoting the integral of the 3-form $-i\mathrm{r}\frac{\partial}{\partial s} A_* \wedge w_*$ over $\{s\} \times Y_*$. Only a portion of the integral of $-i\mathrm{r}\frac{\partial}{\partial s} A_* \wedge w_*$ contributes to $\mathfrak{i}_w$. To say more, write A_* as $A_E + \hat{\mathfrak{a}}_A$ with $\hat{\mathfrak{a}}_A$ denoting an s-dependent 1-form on Y_*. The integral of the 3-form $-i\mathrm{r}\frac{\partial}{\partial s} A_* \wedge w_*$ over $\{s\} \times Y_*$ is written using $\hat{\mathfrak{a}}_A$ as

$$-\, i\mathrm{r}\frac{\partial}{\partial s}\Big(\int_{\{s\}\times Y_*} \hat{\mathfrak{a}}_A \wedge w_*\Big) + i\mathrm{r}\Big(\int_{\{s\}\times Y_*} \hat{\mathfrak{a}}_A \wedge \frac{\partial}{\partial s} w_*\Big). \tag{8.45}$$

The contributions of the function W in (2.6) to the right hand side of (8.44) are given by the integral over I_L of the left most term in (8.45), this being a consequence of the fundamental theorem of calculus. What is denoted by $\mathfrak{i}_w$ is the integral over I_L of the right most term in (8.45). A bound for the absolute value of the latter is obtained by using the the assumption in Item b) of the fourth bullet of (3.13) to write $\frac{\partial}{\partial s} w_*$ as $d\mathfrak{b}$ with $\mathfrak{b}$ as described by this same part of (3.13). Stokes' theorem equates the the right most integral in (8.45) with the integral of $i\mathrm{r}\, d\, \hat{\mathfrak{a}}_A \wedge \mathfrak{b}$. This being the case, it follows from (3.13) that this second contribution to $\mathfrak{i}_w$ has absolute value less than $c_0 \mathrm{r}^{2-1/c}$.

Part 3: Integrate $|F_{\mathbb{A}}^+ - \mathrm{r}(\psi^\dagger \tau \psi - i w_X^+) - i\mathfrak{w}_\mu^+|^2 + \mathrm{r}|D_{\mathbb{A}}\psi|^2$ over $s^{-1}([-L+4, L-4])$. Integrate by parts using the fact this integral is zero to derive an identity that can be written as

$$\begin{aligned} &\frac{1}{2}\int_{s^{-1}([-L+4,L-4])} \big(|F_{\mathbb{A}}|^2 + \mathrm{r}^2|\psi^\dagger \tau\psi - i w_X^+|^2 + 2\mathrm{r}|\nabla_{\mathbb{A}}\psi|^2\big) + \mathfrak{i}_L \\ &\qquad = \mathfrak{a}(\mathfrak{d}|_{s=-L+4}) - \mathfrak{a}(\mathfrak{d}|_{s=L-4}) \end{aligned} \tag{8.46}$$

with $\mathfrak{i}_L$ such that $|\mathfrak{i}_L| \le c_0 \mathrm{r}^{1+c_0/c}$. The paragraphs that follow momentarily derive the latter bound. By way of comparison, the absolute value of the term $\mathfrak{i}$ in (8.17) has the bound $c_0 c\mathrm{r}\big(\int_{s^{-1}([-L+4,L-4])} |F_{\mathbb{A}}|^2\big)^{1/2} + c_0\mathrm{r}^{1+c_0/c}$. The difference can be traced to the assumption that w_X is a closed 2-form on $s^{-1}([-L+4, L-4])$.

The bound on $|\mathfrak{i}_L|$ can be seen by writing $\mathfrak{i}_L$ as a sum of four integrals, these denoted by $\mathfrak{i}_\psi$, $\mathfrak{i}_{cs}$, $\mathfrak{i}_w$ and $\mathfrak{i}_\mu$. The integrand of $\mathfrak{i}_\psi$ is $\frac{1}{4}\mathrm{r}|\psi|^2\mathrm{R}$ with R denoting the scalar curvature of X. By way of an explanation, this term comes from the integration by parts and subsequent commuting of covariant derivatives that rewrites the integral of $\mathrm{r}|D_{\mathbb{A}}\psi|^2$ as an integral over the $s^{-1}(-L+4)$ and $s^{-1}(L-4)$ boundaries of the integration domain plus an integral over $s^{-1}([-L+4, L-4])$ whose integrand is the sum of $\mathrm{r}|\nabla_{\mathbb{A}}\psi|^2$, a curvature term involving $F_{\mathbb{A}}^+$ and the product of $\frac{1}{4}\mathrm{r}|\psi|^2\mathrm{R}$ with R denoting the scalar curvature of the metric on X. The boundary terms account for the right most integral in (2.5)'s formula for $\mathfrak{a}$. Use the bounds from the first two bullets of (3.13) with the bound $|\psi|^2 \le c_0 c$ from Lemma 8.1 to see that $|i_\psi| \le c_0\mathrm{r}^{1+2/c}$ if $\mathrm{r} > c_c$ with c_c again denoting a constant that depends only on c.

The integrals i_{cs} and $\mathfrak{i}_w$ involve a chosen Hermitian connection on $\det(\mathbb{S}^+)$ whose curvature has norm bounded by $c\mathrm{r}^{c_0/c}$ and whose pull back from the $s \le -L+8$ and $s \ge L-8$ part of X via the embeddings from the second and third bullets is the respective Y_- and Y_+ versions of $A_K + 2A_E$. Step 3 of the proof of Lemma 8.1 explains why such connections exist. Let $\mathbb{A}_{\mathbb{S}}$ denote a chosen connection with this property.

The integral $\mathfrak{i}_{cs}$ comes by first writing $|F_{\mathbb{A}}^+|^2$ as $\frac{1}{2}|F_{\mathbb{A}}|^2$ plus the Hodge star of $\frac{1}{2}F_{\mathbb{A}} \wedge F_{\mathbb{A}}$. The latter is rewritten using an integration by parts after writing $\mathbb{A}$ as $\mathbb{A}_{\mathbb{S}} + \hat{\mathrm{a}}_A$ with $\hat{\mathrm{a}}_A$ being an $i\mathbb{R}$-valued 1-form on X. Writing $\mathbb{A}$ in this way yields

$$\frac{1}{2}F_{\mathbb{A}} \wedge F_{\mathbb{A}} = \frac{1}{2}d\,\hat{\mathrm{a}}_A \wedge d\,\hat{\mathrm{a}}_A + d\,\hat{\mathrm{a}}_A \wedge F_{\mathbb{A}_{\mathbb{S}}} + \frac{1}{2}F_{\mathbb{A}_{\mathbb{S}}} \wedge F_{\mathbb{A}_{\mathbb{S}}}. \tag{8.47}$$

An integration by parts writes the integrals of the first two terms on the right side of (8.47) as boundary integrals, these giving the respective cs contributions to $\mathfrak{a}(\mathfrak{d}|_{s=L+4})$ and $\mathfrak{a}(\mathfrak{d}|_{s=L-4})$. The integral of the right most term in (8.43) is $\mathfrak{i}_{cs}$. Thus $|\mathfrak{i}_{cs}| \le c_0\mathrm{r}^{c_0/c}$.

The integral $\mathfrak{i}_w$ is obtained by invoking Stokes' theorem to rewrite the term from the inner product between $F_{\mathbb{A}}^+$ and $\frac{i}{2}\mathrm{r}w_X$ that arises when $|F_{\mathbb{A}}^+ - \mathrm{r}(\psi^\dagger\tau\psi - \frac{i}{2}w_X) + i\mathfrak{w}_\mu^+|^2$ is written as $|F_{\mathbb{A}}^+|^2 + \mathrm{r}|\psi^\dagger\tau\psi - w_X|^2$ plus remainder terms. One of these remainder terms is twice the inner product of $F_{\mathbb{A}}^+$ with $\frac{i}{2}\mathrm{r}w_X$. The integral of the latter is the integral of the 4-form $-i\mathrm{r}F_{\mathbb{A}}^+ \wedge w_X$. Write $-i\mathrm{r}F_{\mathbb{A}} \wedge w_X$ as the sum of $-i\mathrm{r}d\,\hat{\mathrm{a}}_A \wedge w_X$ and $-2i\mathrm{r}F_{\mathbb{A}} \wedge w_X$. Because w_X is closed, an integration by parts writes the integral of

the first of these as an integral over the boundary of the integration domain. The latter accounts for the respective W contributions $\mathfrak{a}(\mathfrak{d}|_{s=-L+4})$ and $\mathfrak{a}(\mathfrak{d}|_{s=L-4})$. The integral of $-2i\mathrm{r}F_{\mathbb{A}}\wedge w_X^+$ is $\mathfrak{i}_w$. This being the case, the bound $|\mathfrak{i}_w| \leq c_0 c\mathrm{r}^{1+c_0/c}$ follows directly from the (3.13) and what is said in Step 3 of the proof of Lemma 8.1 about $|F_{\mathbb{A}_\mathbb{S}}|$.

The integral denoted by $\mathfrak{i}_\mu$ has two contributions. The first accounts for the terms with $\mathfrak{w}_\mu$ that arise in the aforementioned rewriting of $|F_{\mathbb{A}}^+ - \mathrm{r}(\psi^\dagger\tau\psi - iw_X) + i\mathfrak{w}_\mu^+|^2$. It follows from the left hand equation in (2.9) that the integrand for this part of $\mathfrak{i}_\mu$ is bounded by c_0. The second contribution is proportional to the integral of $d\,\hat{\mathfrak{a}}_A\wedge\mathfrak{w}_\mu$; it appears when Stokes' theorem is used to write the respective $\mathfrak{e}_\mu$ parts of $\mathfrak{a}(\mathfrak{d}|_{s=-L+4})$ and $\mathfrak{a}(\mathfrak{d}|_{s=L-4})$ as a term that has norm bounded by c_0 and another whose integrand is proportional to $d\,\hat{\mathfrak{a}}_A\wedge\mathfrak{w}_\mu$. The norm of the latter is bounded by $c_0 c(|F_{\mathbb{A}}| + c^2)$. Granted this, it follows that $|\mathfrak{i}_\mu| \leq c_0 c\Big(\big(\int_{s^{-1}([-L+4,L-4])}|F_{\mathbb{A}}|^2\big)^{1/2} + c^2\Big)$ and this is guaranteed by Lemma 8.1 to be less than $c_0 c(\mathrm{r}+c^2)$.

Part 4: If the first bullet of (3.14) holds, the assumption $\mathfrak{a}(\mathfrak{c}_-)-\mathfrak{a}(\mathfrak{c}_+) < \mathrm{r}^{2-1/c}$ with (8.16) and (8.43) imply that

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}\times Y_Z}\Big(|\frac{\partial}{\partial s}A_*|^2 + |\mathfrak{B}_{\mathfrak{d}}|^2 + 2\mathrm{r}(|\frac{\partial}{\partial s}\psi_*|^2 + |D_A\psi|^2)\Big)\\
&\quad\leq \mathfrak{a}(\mathfrak{c}_-) - \mathfrak{a}(\mathfrak{c}_+) + c_0\mathrm{r}\\
&\quad\leq c_0\mathrm{r}^{2-1/c}
\end{aligned}
\tag{8.48}
$$

when $c > c_0$ and r is greater than a constant that depends only on c. If the second bullet of (3.14) holds, the assumption $\mathfrak{a}(\mathfrak{c}_-)-\mathfrak{a}(\mathfrak{c}_+) < \mathrm{r}^{2-1/c}$ with (8.16), (8.44) and (8.46) imply the bounds that follow when $c > c_0$ and r is greater than a constant that depends only on c:

$$
\begin{cases}
\int_{(-\infty,-L+4]\times Y_-}\Big(|\frac{\partial}{\partial s}A_*|^2 + |\mathfrak{B}_{\mathfrak{d}}|^2 + 2\mathrm{r}(|\frac{\partial}{\partial s}\psi_*|^2 + |D_A\psi|^2)\Big) \leq c_0\mathrm{r}^{2-1/c}.\\
\int_{[L-4,\infty)\times Y_+}\Big(|\frac{\partial}{\partial s}A_*|^2 + |\mathfrak{B}_{\mathfrak{d}}|^2 + 2\mathrm{r}(|\frac{\partial}{\partial s}\psi_*|^2 + |D_A\psi|^2)\Big) \leq c_0\mathrm{r}^{2-1/c}.\\
\int_{s^{-1}([-L+4,L-4])}\Big(|F_{\mathbb{A}}|^2 + \mathrm{r}^2|\psi^\dagger\tau\psi - iw_X^+|^2 + 2\mathrm{r}|\nabla_{\mathbb{A}}\psi|^2\Big) \leq c_0\mathrm{r}^{2-1/c}.
\end{cases}
\tag{8.49}
$$

Put away for now the bounds in (8.48) and those in the first two bullets of (8.49). Assuming that the second bullet of (3.14) holds, the bound in the third bullet of (8.49) implies the bound

$$
\mathrm{r}\int_{s^{-1}([-L+4,L-4])}|\psi^\dagger\tau\psi - iw_X^+| \leq c_0\mathrm{r}^{1-1/c}
\tag{8.50}
$$

when $c > c_0$ and r is greater than a constant that depends only on c. Let B denote the given ball from Lemma 8.5. Use the second and third bullets of (2.8) and (3.12), the

first bullet of Lemma 8.2, and (8.50) to see that

$$\mathrm{r}\int_{B\cap s^{-1}([-L+4,L-4])} |1-|\alpha|^2| \leq c_0\mathrm{r}^{1-1/c}, \tag{8.51}$$

when r is greater than a purely c-dependent constant.

Part 5: If the first bullet of (3.14) holds, then I denotes in what follows any given length 1 interval in $\mathbb{R}$. If the second bullet of (3.14) holds, then I denotes a length 1 interval in either $(-\infty, -L+4]$ or in $(L-4, \infty)$. In either case, reintroduce the 1-form υ_X from the fifth bullet of (3.14). Take the inner product of both sides of (8.41) with $i\upsilon_X$; then integrate the resulting identity over $s^{-1}(I)$. The left hand side of the result can be written as a sum of four integrals; and the assertion that this sum is zero can be rewritten as the identity

$$\begin{aligned}\int_I\Big(\int_{Y_*}\upsilon_X\wedge \mathrm{r}(w_*+*i\psi^\dagger\tau\psi)\Big)ds &= \int_I\Big(\int_{Y_*}\upsilon_X\wedge id\,\hat{\mathrm{a}}_{\mathbb{A}}\Big)ds \\ &+\int_I\Big(\int_{Y_*}\upsilon_X\wedge *\frac{\partial}{\partial s}A_*\Big)ds+\int_I\Big(\int_{Y_*}\upsilon_X\wedge *(-\mathfrak{w}_\mu^+(\frac{\partial}{\partial s},\cdot)+\frac{1}{2}iB_{\mathbb{A}_{\mathbb{S}}})\Big)ds.\end{aligned} \tag{8.52}$$

Use what is said by either the first bullet in (3.14) or the second and fifth bullets of (3.13) to bound the absolute value of the right most integral in (8.52) by a purely c-dependent constant. Meanwhile, Stokes' theorem finds the middle integral on the right hand side of (8.52) equal to zero. The absolute value of the left most integral on the right hand side of (8.52) is bounded by $c_0 c$ times the L^2-norm over $s^{-1}(I)$ of $\frac{\partial}{\partial s}A_*$. This being the case, use either (8.48) or the first two bullets in (8.49) to bound the absolute value of the left most integral on the right side of (8.52) by $\mathrm{r}^{1-1/(2c)}$ when r is greater than a purely c -dependent constant.

It follows as a consequence of what was just said in the preceding paragraph that the absolute value of the integral on the left hand side of (8.52) is no greater than $\mathrm{r}^{1-1/(2c)}$ when r is large. The plan for what follows is to rewrite this integral as the sum of two terms, one being the integral of $\mathrm{r}|\upsilon_X||\,|w_*|-|\psi|^2|$ and the other bounded by r^{1-1/c_c}. This is done in Part 7. Part 6 supplies the necessary tools. A bound of this sort with the second and third bullets of (2.8) and (3.12) plus the first bullet of Lemma 8.2 leads directly to the bound

$$\mathrm{r}\int_{B\cap s^{-1}(I)} |1-|\alpha|^2| \leq c_0\mathrm{r}^{1-1/c} \tag{8.53}$$

when B is any given ball from Lemma 8.5. This bound implies Lemma 8.5's assertion if the first bullet of (3.14) holds. This bound with (8.52) imply Lemma 8.5's bound when the second bullet of (3.14) holds.

Part 6: The two lemmas that are stated momentarily and then proved supply what is needed for Part 7. To set the stage for the first lemma, note that Clifford multiplication by w_X splits $\mathbb{S}^+$ where $w_X \neq 0$ as a direct sum of eigenbundles for the endomorphism given by Clifford multiplication by w_X. Write this direct splitting as $\mathbb{S}^+ = E_X \oplus (E_X \otimes K_X^{-1})$ with it understood that the left most summand is the $i|w_X|$-eigenspace. The upcoming lemma writes a section ψ of $\mathbb{S}^+$ where $w_X \neq 0$ as $|w_X|^{1/2}\eta$ and it writes η with respect to the direct sum decomposition of $\mathbb{S}^+$ as (α, β). The lemma that follows asserts bounds for $|\alpha|$ and $|\beta|$ that are the analogs of those asserted by the first two bullets of Lemma 7.2.

Lemma 8.6 *There exists $\kappa > 100$, and given $c \geq \kappa$, there exists κ_c with the following significance: Fix $\mathrm{r} \geq \kappa_c$ and assume that the metric obey the $(c, r = \mathrm{r})$ version of the constraints in the first three bullets of (3.13) and $|w_X| \leq c$, or that the first bullet of (3.14) holds. Fix elements μ_- and μ_+ from the respective Y_- and Y_+ versions of Ω with $\mathcal{P}$-norm bounded by 1 and use all of this data to define the equations in (2.9). Let $\mathfrak{d} = (\mathbb{A}, \psi)$ denote an instanton solution to these equations. Fix $m > 1$. Then*

$$|\alpha|^2 \leq 1 + \kappa_c m^3 \mathrm{r}^{-1+\kappa/c} \quad \textit{and} \quad |\beta|^2 < \kappa m^3 \mathrm{r}^{-1+\kappa/c}(1 - |\alpha|^2) + \kappa^3 m^6 \mathrm{r}^{-2+\kappa/c}$$

at the points in X where $|w_X| > m^{-1}$.

Proof. The proof is much like that of the first two bullets in Lemma 7.2 with the only salient difference being the r-dependent bounds for the norms of the Riemannian curvature and the covariant derivatives of w_X. The paragraphs that follow briefly explain how this r-dependence is dealt with.

The section $\eta = (\alpha, \beta)$ of $\mathbb{S}^+$ obeys an equation of the form $\mathcal{D}_{\mathbb{A}}\eta + \mathfrak{R} \cdot \eta = 0$ with $\mathfrak{R}$ being an endomorphism that is bounded by $c_c m^{-1} \mathrm{r}^{1/c}$ on U_{2m} . The Weitzenböck formula for the operator $(\mathcal{D}_{\mathbb{A}} + \mathfrak{R})^2$ leads to an equation for η that has the schematic form

$$\nabla_{\mathbb{A}}^\dagger \nabla_{\mathbb{A}}\eta - \frac{1}{2}\,\mathrm{cl}(F_{\mathbb{A}}^+)\eta + \mathfrak{R}_1 \cdot \nabla_{\mathbb{A}}\eta + \mathfrak{R}_0 \cdot \eta = 0, \tag{8.54}$$

where $|R_1| \leq c_c m^{-1}$ and $|R_0| \leq c_c m^{-2}$. As in the proof of Lemma 7.2, introduce q to denote the maximum of 0 and $|\eta|^2 - 1$. It follows from (8.54) that q obeys the inequality

$$d^\dagger dq + \mathrm{r} m^{-1} q \leq c_c m^{-2} \mathrm{r}^{2/c} \tag{8.55}$$

on U_{2m} when $\mathrm{r} \geq c_c$. It follows from Lemma 8.1 that $q \leq c_c m$ on the boundary of U_{2m}. This understood, the comparison principle using the Green's function for

$d^\dagger d + \mathrm{r}m^{-1}$ can be used to see that $q - c_c m^3 \mathrm{r}^{-1+2/c}$ is no greater than $c_c m e^{-\sqrt{\mathrm{r}/(2m)}}$ on U_{2m}. This bound on q implies what is said by Lemma 8.6 about $|\alpha|^2$.

To see about the bound for $|\beta|^2$, project (8.54) to the $E_X \otimes K_X^{-1}$ summand of $\mathbb{S}^+$ to see that $|\beta|^2$ obeys a differential inequality on U_{2m} that has the schematic form

$$d^\dagger d|\beta|^2 + \mathrm{r}m^{-1}|\beta|^2 \leq -2|\nabla\beta|^2 + c_k \mathrm{r}^{-1+c_0/c} m^3 |\nabla_A \alpha|^2 + c_0 m^2 \mathrm{r}^{c_0/c} \tag{8.56}$$

when $\mathrm{r} \geq c_c$. Meanwhile, the projection of (8.54) to the E_X summand can be used to see that $\mathrm{w} = 1 - |\alpha|^2$ on U_{2m} obeys the following analog of any given $\varepsilon > 0$ version of (7.5)

$$d^\dagger d\mathrm{w} + \mathrm{r}m^{-1}\mathrm{w} \geq |\nabla\alpha|^2 - c_0\varepsilon|\nabla\beta|^2 - c_0(1+\varepsilon^{-1})m^2\mathrm{r}^{c_0/c}. \tag{8.57}$$

It follows from (8.56) and (8.57) that there are constants z_1 and z_2 that are both bounded by c_c, and there exists an $\varepsilon > c_c^{-1}$ such that $q := |\beta|^2 - z_1 \mathrm{r}^{-1+c_0/c} m^3 \mathrm{w} - z_2 \mathrm{r}^{-2+c_0/c} m^6$ obeys the equation $d^\dagger dq + \mathrm{r}m^{-1}q \leq 0$ on U_{2m}. This being the case, a comparison principle argument much like that used in the preceding paragraph bounds q by $c_c m e^{-\sqrt{\mathrm{r}/2m}}$ on U_{2m}. This bound implies Lemma 8.6's assertion about $|\beta|^2$. □

The next lemma supplies an analog for X of Lemma 7.3.

Lemma 8.7 *There exists $\kappa > 100$, and given $c \geq \kappa$, there exists κ_c with the following significance: Fix $\mathrm{r} \geq \kappa_c$ and assume that the metric obeys the $(c, r = \mathrm{r})$ version of the constraints in the first three bullets of (3.13) and $|w_X| \leq c$, or that the first bullet of (3.14) holds. Fix elements μ_- and μ_+ from the respective Y_- and Y_+ versions of Ω with $\mathcal{P}$-norm bounded by 1 and use this data to define the equations in (2.9). Let $\mathfrak{d} = (\mathbb{A}, \psi)$ denote an instanton solution to these equations. Fix $m > 1$. Then $|\psi|^2 \leq \kappa_c(m^{-1} + c_c \mathrm{r}^{-1+\kappa/c})$ at points in X where $|w_X| \leq m^{-1}$.*

Proof. The Weitzenböck formula for $\mathcal{D}_{\mathbb{A}}^2$ was used in Step 1 of the proof of Lemma 8.1 to write the differential inequality $d^\dagger d|\psi| + \mathrm{r}(|\psi|^2 - |w_X| - c_c\mathrm{r}^{-1+1/c})|\psi| \leq 0$. The maximum principle precludes a local maximum for $|\psi|^2 - m^{-1} - c_c\mathrm{r}^{-1+1/c}$ on $X - U_m$ and Lemma 8.6 implies that $|\psi|^2 \leq 2(m^{-1} + c_c m^2 \mathrm{r}^{-1+c_0/c})$ on the boundary of $X - U_m$. □

Part 7: Fix $m > 1$ for the moment and write $(ds \wedge \upsilon_X)^+$ on U_m as $q_X w_X + \mathfrak{b}_X$ with $\mathfrak{b}_X$ being a self-dual 2-form that obeys $\mathfrak{b}_X \wedge w_X = 0$. Note in this regard that

$$q_X|w_X|^2 = *(ds \wedge \upsilon_X \wedge w_X) \tag{8.58}$$

with the $*$ here denoting the Hodge star that is defined by the metric $ds^2 + \mathfrak{g}$ on $I \times Y_*$. Granted (8.58), it follows either from the first bullet of (3.14) or from the fourth bullet and Item c) of the fifth bullet of (3.13) that

$$q_X |w_X|^2 \geq -c_c \mathrm{r}^{-1/c}. \tag{8.59}$$

Noting that $*(ds \wedge \upsilon_X \wedge w_X)$ is also the $\mathfrak{g}$-Hodge star on Y_* of $\upsilon_X \wedge w_*$, the integrand of the U_m part of the integral on the left hand side of (8.52) is

$$\mathrm{r} q_X |w_*|^2 (1 - |\alpha|^2 + |\beta|^2) + \mathfrak{r} \quad \text{where } |\mathfrak{r}| \leq c_c \mathrm{r} |\mathfrak{b}| |w_X| |\alpha| |\beta|. \tag{8.60}$$

Use the bound in (8.59) and the bounds supplied by Lemma 8.6 to see that the U_m part of the integral on the left side of (8.52) can be written as

$$\mathrm{r} \int_{U_m} |q_X| |w_*| ||w_*| - |\psi|^2| + \mathfrak{e} \quad \text{where } |\mathfrak{e}| \leq c_c (\mathrm{r}^{1-c_0/c} + m^3 \mathrm{r}^{c_0/c}). \tag{8.61}$$

Meanwhile, it follows from Lemma 8.7 that the contribution to the integral on the left side of (8.52) from $X - U_m$ is no greater than $c_c(\mathrm{r}m^{-1} + m^2 \mathrm{r}^{c_0/c})$. Lemma 8.7 also gives such a bound for the integral of $|q_X| |w_*| ||w_*| - |\psi|^2|$ over the part of $I \times Y_*$ in $X - U_m$. Granted these bounds, fix for the moment $\varepsilon > 0$ but with $\varepsilon < c_0 c^{-1}$ and take $m = \mathrm{r}^{\varepsilon/c}$. Use the just stated bounds and (8.61) to see that

$$\int_{I \times Y_*} |q_X| \, |w_*| \, ||w_*| - |\psi|^2| \leq \int_{I \times Y_*} ds \wedge \upsilon_X \wedge \mathrm{r}(w_* + *i\psi^\dagger \tau \psi) + c_c \mathrm{r}^{1-\varepsilon/c}. \tag{8.62}$$

This last bound with what is said at the end of Part 5 implies Lemma 8.5. □

8.7 Proof of Proposition 3.7

Fix a smooth, r-independent metric on X whose pull-back via the embeddings from the second and third bullets of (2.7) restricts to the $s < -2$ and $s > 2$ parts of X as the product metric $ds^2 + \mathfrak{g}_*$, where $\mathfrak{g}_*$ denotes the given metric on Y_- and Y_+ as the case may be. Use $\mathfrak{m}_X$ to denote this metric. Use this metric to define the bundles $\mathbb{S}^+$ and $\mathbb{S}^-$ over X. The constructions at the beginning of Section 8.3 can be repeated to view the Y_- and Y_+ version of $\mathbb{S}$ as the restrictions to the respective $s < -1$ and $s > 1$ parts of X of the $\mathfrak{m}_X$ versions of $\mathbb{S}^+$ and $\mathbb{S}^-$. Use this view of these versions of $\mathbb{S}$ to view the Y_- and Y_+ versions $A_K + 2A_E$ as a Hermitian connection on the restriction of the $\mathfrak{m}_X$ version of the bundle $\det(\mathbb{S}^+)$ to the $|s| > 1$ part of X. This connection has smooth, r-independent extensions to the whole of X as a Hermitian connection on the $\mathfrak{m}_X$ version of $\det(\mathbb{S}^+)$. Fix such an extension and denote it by $\mathbb{A}_{\mathbb{S}}$.

Use the $s < -1$ and $s > 1$ isomorphisms between the Y_- and Y_+ versions of $\mathbb{S}$ to view the corresponding versions of ψ_E as a section of the $\mathfrak{m}_X$ version of $\mathbb{S}^+$ over the $|s| > 1$ part of X. Fix a smooth extension of the latter to the whole of X and denote it by $\psi_{\mathbb{S}}$.

The metric $\mathfrak{m}_X$ and the pair $\mathfrak{d}_{\mathbb{S}} = (\mathbb{A}_{\mathbb{S}}, \psi_{\mathbb{S}})$ defines a version of the operator that appears in (2.61) of [T3]. This operator defines a map from $C^\infty(X; iT^*X \oplus \mathbb{S}^+)$ to $C^\infty(X; \Lambda^+ \oplus \mathbb{S}^- \oplus i\mathbb{R})$. The latter defines an unbounded, Fredholm operator between the L^2-versions of these spaces, and so it has a corresponding Fredholm index, this denoted in what follows as $\imath_{\mathbb{S}}$.

Fix $c > c_0$ so that Proposition 3.6 can be invoked using Y_- and Y_+. Fix $\mathrm{r} \gg 1$ and pairs μ_- and μ_+ from the respective Y_- and Y_+ versions of Ω with $\mathcal{P}$-norm less than 1; and suppose that $\mathfrak{c}_-$ and $\mathfrak{c}_+$ are the corresponding solutions to the Y_- and Y_+ versions of (2.4). Let $\mathfrak{m}$ denote a metric on X that obeys (2.8) and (3.12). Suppose that $\mathfrak{d} = (\mathbb{A}, \psi)$ is a pair of connection on $\det(\mathbb{S}^+)$ over X and section over X of $\mathbb{S}^+$ with $s \to -\infty$ limit $\mathfrak{c}_-$ and $s \to \infty$ limit $\mathfrak{c}_+$. This metric $\mathfrak{m}$ and $\mathfrak{d}$ together define a corresponding version of the operator that appears in (2.61) of [T3]. If both $\mathfrak{c}_-$ and $\mathfrak{c}_+$ are non-degenerate then this operator has an unbounded, Fredholm extension whose domain and range are the respective spaces of square integrable sections of $iT^*X \oplus \mathbb{S}^+$ and $i\Lambda^+ \oplus \mathbb{S}^- \oplus i\mathbb{R}$. Assume this to be the case for the moment, and let $\imath_{\mathfrak{d}+}$ denote the corresponding Fredholm index. It follows using the excision theorem for the index (or from what is said in [APS]) that $\imath_{\mathbb{S}} = \imath_{\mathfrak{d}+} + \mathfrak{f}_s(\mathfrak{c}_-) - \mathfrak{f}_s(\mathfrak{c}_+)$.

With the preceding understood, write $\mathfrak{a}(\mathfrak{c}_-) - \mathfrak{a}(\mathfrak{c}_+)$ as

$$\mathfrak{a}^{\mathfrak{f}}(\mathfrak{c}_-) - \mathfrak{a}^{\mathfrak{f}}(\mathfrak{c}_+) - 2\pi(\mathrm{r} - \pi)(\mathfrak{f}_s(\mathfrak{c}_-) - \mathfrak{f}_s(\mathfrak{c}_+)) \tag{8.63}$$

and then use the formula in the last paragraph to write

$$\mathfrak{a}(\mathfrak{c}_-) - \mathfrak{a}(\mathfrak{c}_+) = \mathfrak{a}^{\mathfrak{f}}(\mathfrak{c}_-) - \mathfrak{a}^{\mathfrak{f}}(\mathfrak{c}_+) + 2\pi(\mathrm{r} - \pi)(\imath_{\mathfrak{d}+} - \imath_{\mathbb{S}}). \tag{8.64}$$

Since $\imath_{\mathbb{S}}$ is independent of r and c, this last formula proves Proposition 3.7 when both $\mathfrak{c}_-$ and $\mathfrak{c}_+$ are non-degenerate.

If one or neither is non-degenerate, fix $\varepsilon > 0$ and fix $\mathfrak{c}'_-$ in the set $\mathfrak{N}_\varepsilon(\mathfrak{c}_-)$ from Section 7.6 that takes on the supremum in the $\mathfrak{c}_-$ version of (7.31). Fix c'_+ in $\mathfrak{N}_\varepsilon(c_+)$ with the analogous property. With $\mathfrak{c}'_-$ and $\mathfrak{c}'_+$ as just described, choose a pair $\mathfrak{d}'$ of connection on $\det(\mathbb{S}^+)$ and section of $\mathbb{S}^+$ with $s \to -\infty$ limit $\mathfrak{c}'_-$ and $s \to \infty$ limit $\mathfrak{c}'_+$. The metric $\mathfrak{m}$ with $\mathfrak{d}'$ define an unbounded, but now Fredholm version of the operator from (2.62) in [T3] with domain and range being the respective spaces of square integrable sections of $iT^*X \oplus \mathbb{S}^+$ and $i\Lambda^+ \oplus \mathbb{S}^- \oplus i\mathbb{R}$. Let $\imath_{\mathfrak{d}'}$ denote the Fredholm index of this

operator. Define $\imath_{\mathfrak{d}+}$ to be $\imath_{\mathfrak{d}'+}$. Note that this definition does not depend on $\mathfrak{c}'_-$, $\mathfrak{c}'_+$ or $\mathfrak{d}'$.

The arguments that lead to (8.64) can be repeated verbatim to obtain the modified version that has $\mathfrak{c}_-$ replaced by $\mathfrak{c}'_-$ and $\mathfrak{c}_+$ replaced by $\mathfrak{c}'_+$. Keeping this in mind, choose $\mathfrak{c}'_-$ so that $|\mathfrak{a}(\mathfrak{c}'_-) - \mathfrak{a}(\mathfrak{c}_-)| < 1$, and choose $\mathfrak{c}'_+$ so that $|\mathfrak{a}(\mathfrak{c}'_+) - \mathfrak{a}(\mathfrak{c}_+)| < 1$. It follows using (7.31) that $|\mathfrak{a}^{\mathfrak{f}}(\mathfrak{c}'_-) - \mathfrak{a}^{\mathfrak{f}}(\mathfrak{c}_-)| < 1$ and $|\mathfrak{a}^{\mathfrak{f}}(\mathfrak{c}'_+) - \mathfrak{a}^{\mathfrak{f}}(\mathfrak{c}_+)| < 1$. The latter bound with the $(\mathfrak{c}'_-, \mathfrak{c}'_+)$ analog of (8.64) implies what is asserted by Proposition 3.7 when the non-degeneracy condition does not hold for one or both of $\mathfrak{c}_-$ and $\mathfrak{c}_+$. □

9 Constructing 2-forms on cobordisms

This section supplies proofs for Propositions 3.8, 3.10, 3.12 and 3.13. The proof of Proposition 3.8 is in Section 9.2, that of Proposition 3.10 is in Section 9.4, that of Proposition 3.12 is in the final subsection, Section 9.5, and Section 9.6 contains the proof of Proposition 3.13. Intervening sections supply the required background material.

9.1 Met_T metrics on $\{Y_k\}_{k\in\{0,\ldots,\mathbf{G}\}}$

The eight parts of this section describe a set of preferred metrics on each $k \in \{0, \ldots, \mathrm{G}\}$ version of Y_k. These parts also describe the associated harmonic 2-forms with de Rham cohomology class that of $c_1(\det(\mathbb{S}))$. Let Y_* denote Y_k for any $k \in \{0, \cdots, \mathrm{G}\}$. As the $M_\delta \cup \mathcal{H}_0$ part of Y_* and Y are canonically isomorphic, notions defined on any of them are defined for others and are denoted by the same notation.

Part 1: This part of the subsection summarizes various properties of Y_* that concern $\mathcal{H}_0$ and the curve $\gamma^{(z_0)}$. Most of what is said below can be found in Section II.1.

The handle $\mathcal{H}_0$ in Y_* has coordinates (u, θ, ϕ) with (θ, ϕ) being the standard spherical coordinates on the 2-sphere and with $u \in [-R - \ln(7\delta_*), R + \ln(7\delta_*)]$. As can be seen in (IV.1.5), the 2-form w and the 1-form $\upsilon_\diamond$ restrict to this handle as

$$w = \sin\theta \, d\theta \wedge d\phi \quad \text{and} \quad \upsilon_\diamond = 2\big(\chi_+ e^{2(|u|-R)} + \chi_- e^{-2(|u|+R)}\big)\, du, \tag{9.1}$$

where $\chi_+ = \chi(-u - \frac{1}{4}R)$ and $\chi_- = \chi(u - \frac{1}{4}R)$. The curve $\gamma^{(z_0)}$ intersects $\mathcal{H}_0$ as the $\theta = 0$ line. Meanwhile, the M_δ part of $\gamma^{(z_0)}$ has a tubular neighborhood with

coordinates $(t,(\theta,\phi))$ with $t \in [\delta, 3-\delta]$, with $\theta \in [0,\theta_*)$ and with ϕ the affine coordinate on $\mathbb{R}/(2\pi\mathbb{Z})$. Here, θ_* is positive, smaller than $\frac{1}{100}\delta_*$ but greater $100\delta^3$. The 2-form w here appears as in (9.1) and $\upsilon_\diamond$ appears as dt. The coordinate transition function identifies t with $e^{-2(R-u)}$ near the index 0 critical point and with $e^{-2(R+u)}$ near the index 3 critical point.

Recall the function f on M that plays a central role in much of [KLT1]-[KLT4]. This is described in detail in Section II.1. Recall also the vector field v in [KLT2] p.19. Set $\varepsilon_* = 3\delta_* \sin(\frac{1}{2}\theta_*)$. The coordinates just described can be used to construct a piecewise smooth embedded 2-sphere in the $f \in [\varepsilon_*^2, 3-\varepsilon_*^2]$ part of M_δ as follows:

(9.2)

- The 2-sphere intersects the complement of the radius-δ_* coordinate balls about the index 0 and 3 critical points of f as the cylinder where $\theta = \frac{1}{2}\theta_*$.
- The 2-sphere intersects the $r \in (\varepsilon_*, \delta_*]$ part of the radius-δ_* coordinate ball centered on the the index 0 and index 3 critical points of f as the locus where (r,θ,ϕ) are such that $\cos\theta > 0$ and $r\sin\theta = 3\delta_* \sin(\frac{1}{2}\theta_*)$.
- The 2-sphere intersects the $r = \varepsilon_*$ spheres centered about the index 0 and index 3 critical points as the locus where $\cos\theta \le 0$.
- The 2-sphere is tangent to v on the rest of M_δ.

As can be seen, this embedding is smooth except along the following loci: It is C^1 on the $\cos\theta = 0$ circle in the boundary of the respective radius ε_* coordinate balls about the index 0 and index 3 critical points of f. It is only C^0 on the $\theta = \frac{1}{2}\theta_*$ circle in the boundary of the respective radius δ_* coordinate balls about the index 0 and index 3 critical points.

The piecewise smooth embedding just described can be smoothed to any desired accuracy so that the vector field $\frac{\partial}{\partial\phi}$ along the resulting 2-sphere is everywhere tangent, the vector field v along the 2-sphere is tangent everywhere on the $f = \frac{3}{2}$ circle but nowhere else, and so that the restriction of f to this sphere has just two critical points, these at the points with $\theta = 0$ and $\theta = \pi$ on the boundary of the radius ε_* coordinate balls about its respective index 0 and index 3 critical points.

Part 2: It proves useful for what follows to be somewhat more precise about the smoothing of the surface from (9.2) near the $f = \frac{3}{2}$ circle. To this end, introduce first ρ_* to denote $\frac{3}{2} - \varepsilon_*^2$ and $\rho_{1*} = \rho_* + \sqrt{2}\big(1 - \cos(\frac{1}{2}\theta_*)\big)$. Return to the $f \in [\delta, 3-\delta]$ tubular neighborhood of $\gamma^{(z_0)}$ with the coordinates $(t,(\theta,\phi))$ as described

above. Replace the coordinate θ on a neighborhood of the $\theta = \frac{1}{2}\theta_*$ locus by the function $\hat{\rho} = \sqrt{2}(1-\cos\theta)^{1/2}$. Fix $\varepsilon_1 \in (0, c_0^{-1}\varepsilon_*^2)$ and use the coordinate $\hat{\rho}$ to define the smoothing of $f \in (\frac{3}{2} - \varepsilon_1, \frac{3}{2} + \varepsilon_1)$ part of the surface defined by (9.2) to be the locus where

$$\hat{\rho} = \rho_1 - \big(\rho_*^2 - (t - \frac{3}{2})^2\big)^{1/2}. \tag{9.3}$$

Note that the vector field v is tangent to the locus defined by (9.3) only along the $t = 0$ circle, and note that the corresponding lines are tangent from the inside. Introduce by way of notation S to denote a smoothing as just described of the original piecewise smooth embedding given by (9.2). (This is the sphere denoted by S_z in [L], about Equation (6.2).)

Part 3: Use (x_1, x_2, x_3) for the Euclidean coordinates on $\mathbb{R}^3$. The function f and the $\mathbb{R}/(2\pi\mathbb{Z})$-valued coordinate function ϕ can be used to embed a neighborhood of S into $\mathbb{R}^3$ as the radius ρ_* sphere about the origin so as to identify $f - \frac{3}{2}$ with x_3 and the vector fields v and $\frac{\partial}{\partial\phi}$ with $\frac{\partial}{\partial x_3}$ and $x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}$. To see how this is done, start by embedding S in $\mathbb{R}^3$ as the sphere of radius $\rho_* = \frac{3}{2} - \varepsilon_*^2$ about the origin by setting $x_3 = f - \rho_*$ and by setting the pair (x_1, x_2) to equal $\Big((2\rho_* f - f^2)^{1/2}\cos\phi, (2\rho_* f - f^2)^{1/2}\sin\phi\Big)$. This embedding is extended to a neighborhood of S by exploiting the fact that the $|f - \frac{3}{2}| > \frac{1}{2}\varepsilon_1$ part of S has a neighborhood with the following property: Let p denote a point in this neighborhood. Then p sits on an integral curve of v that intersects S, and there is precisely one such intersection point with distance $c_\varepsilon^{-1}\varepsilon_1^3$ or less. Here, $c_\varepsilon > 1$ is a constant that depends on ε_1. Such a neighborhood exists because v is tangent to S only on the $f = \frac{3}{2}$ circle in S. Let $\mathcal{N}_1$ denote this neighborhood. Given $p \in \mathcal{N}_1$, let $\eta(p) \in S$ denote the unique point on the integral curve of v through p with distance less than $c_\varepsilon^{-1}\varepsilon_1^3$ from p. Associate to p the point in $\mathbb{R}^3$ with the coordinates

$$x_1(p) = x_1(\eta(p)), \quad x_2(p) = x_2(\eta(p)), \quad x_3(p) = f(p) - \frac{3}{2}. \tag{9.4}$$

To complete the definition of the embedding, suppose next that p is a point near the $f \in (\frac{3}{2} - \varepsilon_1, \frac{3}{2} + \varepsilon_1)$ part of S where the coordinates $(t, \hat{\rho}, \phi)$ are defined. Associate to p the point in $\mathbb{R}^3$ with the coordinates

$$\begin{aligned} x_1(p) &= (\hat{\rho}(p) - \rho_1)^2 \cos\phi(p), \\ x_2(p) &= (\hat{\rho}(p) - \rho_1)^2 \sin\phi(p), \\ x_3(p) &= t(p) - \frac{3}{2}. \end{aligned} \tag{9.5}$$

Note in particular that if p is also in $\mathcal{N}_1$, then it follows from the definition of the function $\hat{\rho}$ and the definition of ρ_1 that the points given by (9.3) and (9.4) are the same.

What is said at the end of the preceding paragraph has the following implication: The map from $\mathcal{N}_1$ to $\mathbb{R}^3$ and the map described in the preceding paragraph together define a smooth, ϕ-equivariant embedding of a neighborhood of S into $\mathbb{R}^3$ that maps S to the radius ρ_* sphere and maps v to $\frac{\partial}{\partial x_3}$.

Fix $\varepsilon > 0$ so that the region in $\mathbb{R}^3$ with $(x_1^2 + x_2^2 + x_3^2)^{1/2} \in (\rho_* - \varepsilon, \rho_* + \varepsilon)$ is in the image of the embedding of $\mathcal{N}_1$. By way of notation, $\mathcal{N}_\varepsilon$ is used in the subsequent discussion to denote both this region in $\mathbb{R}^3$ and its inverse image in M_δ. It is worth keeping in mind for what follows that the points in the $\mathbb{R}^3$ incarnation of $\mathcal{N}_\varepsilon$ with distance *less* than ρ_* from the origin are in the $\mathcal{H}_0$ component of $Y - S$.

By construction, the 1-form $\upsilon_\diamond$ appears on the $\mathbb{R}^3$ version of $\mathcal{N}_\varepsilon$ as dx_3. Meanwhile, the 2-form w must appear here as $\kappa dx_1 \wedge dx_2$ with κ being a strictly positive function of $x_1^2 + x_2^2$. This is because w is closed, it annihilates v and v appears on the $\mathbb{R}^3$ version of $\mathcal{N}_\varepsilon$ as $\frac{\partial}{\partial x_3}$.

Use ρ to denote the function $(x_1^2 + x_2^2)^{1/2}$ on $\mathbb{R}^3$ and introduce the $\mathbb{R}/(2\pi\mathbb{Z})$-valued function ϕ by writing x_1 and x_2 as $\rho\cos\phi$ and $\rho\sin\phi$. The observations from the preceding paragraph, the fact that w is harmonic and the fact that its metric Hodge dual is $\upsilon_\diamond$ have the following implication: The metric from M_δ appears on the $\mathbb{R}^3$ incarnation of $\mathcal{N}_\varepsilon$ as

$$\mathfrak{g} = \kappa\,(h^{-2}d\rho^2 + h^2\rho^2 d\phi^2) + dx_3^2 \tag{9.6}$$

with h denoting a strictly positive function of ρ^2.

Part 4: This part of the subsection says something of the topological significance of S and Part 3's embedding of S and its neighborhood $\mathcal{N}_\varepsilon$ in $\mathbb{R}^3$. To set the stage, recall that Y_0 was obtained from M by attaching the 1-handle $\mathcal{H}_0$. This was done by first deleting the radius $7\delta_*$ coordinate balls about the index 0 and index 3 critical points of f to obtain a manifold with boundary. The resulting boundary spheres were then glued to the $u = R + \ln(7\delta_*)$ and $u = -R - \ln(7\delta_*)$ boundary spheres of $[-R - \ln(7\delta_*), R + \ln(7\delta_*)] \times S^2$.

The sphere S enters a second description of Y_0 as the connected sum of M with the manifold $S^1 \times S^2$. (Cf. [L], (6.2)). The connected sum description constructs Y_0 by deleting the respective 3-balls from M and $S^1 \times S^2$ and gluing the resulting two boundary spheres to the boundary spheres of the product of an interval with S^2. Denote

this product as $I \times S^2$ with $I \subset \mathbb{R}$ being an interval. As explained below, the surface S can be viewed as a cross-sectional sphere of $I \times S^2$.

To see directly this connected sum depiction of Y_0, first view S and $\mathcal{N}_\varepsilon$ as subsets in $\mathbb{R}^3$. Let $r = (\rho^2 + x_3^2)^{1/2}$ denote the radial coordinate on $\mathbb{R}^3$. The connected sum picture of Y_0 results in an embedding of $I \times S^2$ into $\mathbb{R}^3$ whose image is the $r \in [\rho_* - \frac{1}{16}\varepsilon, \rho_* + \frac{1}{16}\varepsilon]$ part of $\mathcal{N}_\varepsilon$. This depiction of $I \times S^2$ in Y_0 identifies the $r = \rho_* + \frac{1}{16}\varepsilon$ sphere in $\mathcal{N}_\varepsilon$ with the boundary of the complement of a ball in $S^1 \times S^2$. This missing ball can be identified with the $r < \rho_* + \frac{1}{16}\varepsilon$ part of $\mathbb{R}^3$. Indeed, the Y_0 incarnation of the $r = \rho_* + \frac{1}{16}\varepsilon$ sphere in $\mathbb{R}^3$ splits Y_0 into two components. The component that contains the $r > \rho_* + \frac{1}{16}\varepsilon$ part of $\mathcal{N}_\varepsilon$ is the complement of a ball in $S^1 \times S^2$; and $S^1 \times S^2$ is reconstituted in full when this complement is filled in by adding the $r \leq \rho_* + \frac{1}{16}\varepsilon$ part of $\mathbb{R}^3$ to the $r > \rho_* + \frac{1}{16}\varepsilon$ incarnation of $\mathcal{N}_\varepsilon$.

The Y_0 incarnation of the $r = \rho_* - \frac{1}{16}\varepsilon$ sphere in $\mathbb{R}^3$ also separates Y_0 into two components. The component that has the $r < \rho_* - \frac{1}{16}\varepsilon$ part of $\mathcal{N}_\varepsilon$ is the complement of a ball in M. This ball is attached to give back M by viewing the complement of its center point as the $r > \rho_* - \frac{1}{16}\varepsilon_*$ part of $\mathbb{R}^3$. To see this, take a second copy of $\mathbb{R}^3$ and use r' to denote the distance to the origin in the latter. Use (θ', φ') to denote the associated spherical coordinates. The manifold M is obtained by attaching the $r' \leq (\rho_* - \frac{1}{16}\varepsilon)^{-1}$ ball in this second copy of $\mathbb{R}^3$ to the $r = \rho_* - \frac{1}{16}\varepsilon$ sphere in the original copy of $\mathbb{R}^3$ via the identifications $r' = r^{-1}$ and $(\theta' = \pi - \theta, \phi' = \phi)$.

Since S splits Y_0 into two parts, it likewise splits Y_* into two parts. The component of $Y_* - S$ that contains $\gamma^{(z_0)}$ has its canonical identification with the $\gamma^{(z_0)}$ component of $Y_0 - S$. The other component of $Y_* - S$ is obtained from the complementary component of $Y_0 - S$ by attaching the $\mathfrak{p} \in \Lambda$ labeled 1-handles.

Both $Y_* - \mathcal{N}_\varepsilon$ and $Y_0 - \mathcal{N}_\varepsilon$ likewise have two components because $\mathcal{N}_\varepsilon$ is a tubular neighborhood of S. A given $k \in \{0, \ldots, \mathrm{G}\}$ version of Y_k is obtained from Y_0 by attaching k 1-handles with attaching regions that are disjoint from the component of $Y_0 - \mathcal{N}_\varepsilon$ that contains $\gamma^{(z_0)}$. This understood, $\mathcal{N}_\varepsilon$ can be viewed as a subset of Y_k and $Y_k - \mathcal{N}_\varepsilon$ also has two components. By way of notation, the component of $Y_* - \mathcal{N}_\varepsilon$ or any given $k \in \{0, \ldots, \mathrm{G}\}$ version of $Y_k - \mathcal{N}_\varepsilon$ that contains $\gamma^{(z_0)}$ is denoted in what follows by $\mathcal{Y}_0$ and the other component is denoted by $\mathcal{Y}_M$. ($\mathcal{Y}_M$ has a natural interpretation as a sutured manifold, which is denoted by $M(1)$ in Remark 1.3).

Part 5: This part of the subsection introduces a family of distinguished metrics on the $k \in \{0, \ldots, \mathrm{G}\}$ version of Y_k that play central roles in the subsequent discussions. Parts 6 and 8 say more about this set.

This distinguished set of metrics is parametrized by a parameter T which is in all cases greater than 1. With T chosen, the corresponding set of metric is denoted in what follows by Met_T . The metrics from Met_T are constructed momentarily from the set of metrics on $\mathcal{Y}_M \cup \mathcal{N}_\varepsilon$ that are given by (9.6) on $\mathcal{N}_\varepsilon$. This set of metrics on $\mathcal{Y}_M \cup \mathcal{N}_\varepsilon$ is denoted by $\mathrm{Met}^{\mathcal{N}}$. Note with regards to (9.6) that its formula depicts a 1-parameter family of metrics with the parameter being the length of the curve $\gamma^{(z_0)}$. The length of $\gamma^{(z_0)}$ plays no role of significance. In any event, the length is assumed to be the same for all metrics in $\mathrm{Met}^{\mathcal{N}}$ whether defined on Y or on a $k \in \{0, \ldots, \mathrm{G}\}$ version of Y_k.

The criteria for membership in Met_T follow directly: All metrics in Met_T agree on $\mathcal{Y}_0 \cup \mathcal{N}_\varepsilon$; the metric they define on this set is denoted in what follows by $\mathfrak{g}_T$. The metric $\mathfrak{g}_T$ on $\mathcal{Y}_0$ is the metric from (3.6). Meanwhile, the metric $\mathfrak{g}_T$ on $\mathcal{N}_\varepsilon$ is defined in the three steps that follow.

Step 1: Introduce χ_r to denote the function on $\mathbb{R}^3$ given by $\chi(64\varepsilon^{-1}(r - \rho_*) - 1)$. This function equals 1 where $r < \rho_* + \frac{1}{64}\varepsilon$ and equals 0 where $r > \rho_* + \frac{1}{32}\varepsilon$. Fix $T > 1$ and introduce r_T to denote $(1 - \chi_r + \frac{1}{T}\chi_r)r$. The r derivative of r_T is strictly positive because that of χ_r is non-positive. Set $\rho_T = r_T \sin\theta$ and $x_{3T} = r_T \cos\theta$. Noting that $d\rho_T$ and dx_{3T} are linearly independent, the quadratic form

$$\mathrm{K}(\rho_T)\,(h^{-2}(\rho_T)\,d\rho_T^2 + h^2(\rho_T)\rho_T^2 d\phi^2) + dx_{3T}^2 \tag{9.7}$$

defines a smooth metric on $\mathbb{R}^3$. The metric $\mathfrak{g}_T$ on the $r > \rho_* - \frac{1}{4}\varepsilon$ part of $\mathcal{N}_\varepsilon$ is given by (9.7).

Step 2: The definition of $\mathfrak{g}_T$ on the $r \in [\rho_* - \frac{1}{2}\varepsilon, \rho_* - \frac{1}{4}\varepsilon]$ part of $\mathcal{N}_\varepsilon$ requires yet another function of r. This one is defined by the rule $r \mapsto \chi(4\varepsilon^{-1}(r - \rho_*) + 2)$ and it is denoted by χ_{r*}. The function χ_{r*} is equal to 1 where $r < \rho_* - \frac{1}{2}\varepsilon$ and it is equal to 0 where $r > \rho_* - \frac{1}{4}\varepsilon$. Set x_{3T*} to denote the function $\left(1 - \chi_{r*} + \frac{1}{T}\chi_{r*}\right) x_3$. Introduce by way of notation K_T and h_T to denote the functions $\mathrm{K}(\rho/T)$ and $h(\rho/T)$. Noting that dx_1, dx_2 and dx_{3T} are linearly independent, the quadratic form

$$\frac{1}{T^2}\mathrm{K}_T(h_T^{-2}d\rho^2 + h_T^2 d\phi^2) + \frac{1}{T^2}dx_{3T*}^2 \tag{9.8}$$

defines a smooth metric on the $r \in [\rho_* - \frac{1}{2}\varepsilon, \rho_* - \frac{1}{4}\varepsilon]$ part of $\mathcal{N}_\varepsilon$. The latter extends the metric given in (9.7) because $\rho_T = \frac{1}{T}\rho$ and $x_{3T} = \frac{1}{T}x_3$ where $\rho < \rho_* + \frac{1}{64}\varepsilon$.

Step 3: The definition of $\mathfrak{g}_T$ on the $r < \rho_* - \frac{1}{2}$ part of $\mathcal{N}_\varepsilon$ requires one more function of r. This one is denoted by χ_{r**} and it is defined by the rule $r \mapsto \chi(4\varepsilon^{-1}(r-\rho_*)+3)$. This function is equal to 0 where $r > \rho_* - \frac{1}{2}\varepsilon$ and it is equal to 1 where $r < \rho_* - \frac{3}{4}\varepsilon$. With this function in hand, define the function T_* to be $T(1-\chi_{r**}) + \chi_{r**}$. The function T_* is equal to T where $r > \rho_* - \frac{1}{2}\varepsilon$ and it is equal to 1 where $r < \rho_* - \frac{3}{4}\varepsilon$. The metric $\mathfrak{g}_T$ is defined on the $r \le \rho_* - \frac{1}{2}\varepsilon$ part of $\mathcal{N}_\varepsilon$ to be the quadratic form

$$\frac{1}{T_*^2}\kappa_{T_*}(h_{T_*}^{-2}d\rho^2 + h_{T_*}^2 d\phi^2) + \frac{1}{4}dx_3^2. \tag{9.9}$$

This definition of $\mathfrak{g}_T$ smoothly extends the metric defined in (9.8). Moreover, the metric $\mathfrak{g}_T$ as just defined is the metric in (9.6) where $r < \rho_* - \frac{3}{4}\varepsilon$.

Part 6: This part of the subsection and Part 8 point out some key properties of the Met_T metrics. This part focuses on the metric $\mathfrak{g}_T$, this being the restriction of each Met_T metric to $\mathcal{Y}_0 \cup \mathcal{N}_\varepsilon$. As explained in the subsequent two paragraphs, each $T > 1$ version of $\mathfrak{g}_T$ on the complement in $\mathcal{Y}_0 \cup \mathcal{N}_\varepsilon$ of the $r \le \rho_*$ part of $\mathcal{N}_\varepsilon$ can be viewed as the pull-back of a T-independent metric on $S^1 \times S^2$ by a T-dependent embedding of the $\gamma^{(z_0)}$ component of $Y_* - S$ or $Y - S$ as the case may be. The embedding is denoted by Φ_T.

To define this T-independent metric on $S^1 \times S^2$, view $S^1 \times S^2$ as in Part 4. By way of a reminder, this view comes with a distinguished ball with a distinguished diffeomorphism onto the $r < \rho_* + \frac{1}{16}\varepsilon$ ball in $\mathbb{R}^3$ centered on the origin. There is in addition, a distinguished identification between the complement of the concentric $r \le \rho_*$ ball in $S^1 \times S^2$ and the union of $\mathcal{Y}_0$ and the $r \ge \rho_*$ part of $\mathcal{N}_\varepsilon$. The latter identifies the metric from Section 1 on $\mathcal{Y}_0$ with a metric on $S^1 \times S^2$ whose restriction to the $r \le \rho_* + \frac{1}{16}\varepsilon$ ball in the distinguished coordinate chart appears as $\kappa(\rho)(h^{-2}(\rho)d\rho^2 + h^2(\rho)\rho^2 d\phi^2) + dx_3^2$. This is the desired T-independent metric on $S^1 \times S^2$. This $S^1 \times S^2$ metric is denoted by $\mathfrak{g}_*$.

Fix $T \ge 1$. The promised embedding of the $\mathcal{Y}_0$ component of $Y_* - S$ into $S^1 \times S^2$ is defined as follows: This embedding agrees with the embedding from the preceding paragraph on $\mathcal{Y}_0$ and on the $r > \rho_* + \frac{1}{32}\varepsilon$ part of $\mathcal{N}_\varepsilon$. Meanwhile, the promised embedding on the $r \in (\rho_*, \rho_* + \frac{1}{16}\varepsilon)$ part of $\mathcal{N}_\varepsilon$ maps the latter onto the $r \in (T^{-1}\rho_*, \rho_* + \frac{1}{16}\varepsilon)$ ball in the distinguished coordinate chart. The map here sends the point with spherical coordinates (r, θ, ϕ) to that with the spherical coordinates (r_T, θ, ϕ).

Part 7: This part of the subsection describes a certain closed 2-form on a given $k \in \{0, \dots, \mathrm{G}\}$ version of Y_k with compact support in $\mathcal{Y}_M$ and with the following

additional property: The de Rham class of this 2-form annihilates all but the $H_2(M;\mathbb{Z})$ summand in the Mayer-Vietoris direct sum decomposition for $H_2(Y;\mathbb{Z})$ in (IV.14) or in the analogous direct sum decomposition for $H_2(Y_k;\mathbb{Z})$. Meanwhile, it acts on the $H_2(M;\mathbb{Z})$ summand as $c_1(\det(\mathbb{S}))$. A version of this 2-form is also defined on M. In all cases, the 2-form is denoted by p. It is used in the upcoming Lemma 9.1 and in later subsections. The construction of p follows directly.

View M_δ as being a subset of each $k \in \{0, \ldots, \mathrm{G}\}$ version of Y_k. As such, it sits in the $\mathcal{Y}_M$ part of Y_k. It follows from the description of $H_2(Y;\mathbb{Z})$ in Part 4 of Section II.1c that there exists a finite set of the form Θ whose elements are pairs of the form $(\gamma, \mathrm{z}_\gamma)$, with γ being a loop in a level set of M_δ of the function f on M. Meanwhile, z_γ is an integer. The loops from Θ generate the image in any given $k \in \{0, \ldots, \mathrm{G}\}$ version of $H_1(Y_k;\mathbb{Z})/\operatorname{tors}$ of $H_1(M;\mathbb{Z})/\operatorname{tors}$ via the Mayer-Vietoris homomorphism for the Y_k analog of the direct sum decomposition in (IV.1.4). Meanwhile, the paired integers are such that $\sum_{\gamma\in\Theta} \mathrm{z}_\gamma\gamma$ represents the image of the Poincaré dual of the restriction of $c_1(\det(\mathbb{S}))$ to the $H_2(M;\mathbb{Z})$ summand in this same direct sum decomposition. Let $(\gamma, \mathrm{z}_\gamma)$ denote a pair from Θ. The loop γ has a tubular neighborhood in M_δ which is the image via an embedding of $S^1 \times D$ where $D \subset \mathbb{R}^2$ is a small radius disk about the origin and where γ corresponds to the image of $S^1 \times \{0\}$. Use $\mathcal{T}_\gamma$ in what follows to denote a tubular neighborhood of this sort. These are to be chosen so that the pairwise distinct versions have disjoint closure that is disjoint from the boundary of the closure of the M_δ part of $\mathcal{N}_\varepsilon$.

Note that there exists such a tubular neighborhood with an embedding that has the following property: The pull back of df via the embedding is a constant 1-form from the D factor of $S^1 \times D$ and the kernel of the pull back via the embedding of the 2-form w is a constant vector field that is tangent to this D factor. The existence of such an embedding follows from two facts, the first being that γ is in an f-level set. The second fact follows from the definition in the first bullet of (IV.1.3) of w on $\mathcal{T}_\gamma$ as the area form for the f-level sets. An embedding of this sort is used in Part 7 of the upcoming Section 9.5.

Fix a compactly supported 2-form on D whose integral is equal to 1. View this 2-form first as an S^1-independent form on $S^1 \times D$ and then as a 2-form on M and on each $k \in \{0, \ldots, \mathrm{G}\}$ versions of Y_k with compact support in $\mathcal{T}_\gamma$. Use p_γ to denote the latter incarnation; then set $p = \sum_{(\gamma,\mathrm{z}_\gamma)\in\Theta} \mathrm{z}_\gamma p_\gamma$. By construction, the de Rham class of p agrees with $c_1(\det(\mathbb{S}))$ on the $H_2(M;\mathbb{Z})$ summand of the Mayer-Vietoris direct sum decomposition of $H_2(Y;\mathbb{Z})$ in (IV.1.4) or its analog for $H_2(Y_0;\mathbb{Z})$ as the case may be. The de Rham class of p also annihilates the $H_2(\mathcal{H}_0;\mathbb{Z})$-summand in these direct sum decompositions. In the case of $H_2(Y;\mathbb{Z})$, the de Rham class of p also annihilates the

$\bigoplus_{\mathfrak{p}\in\Lambda} H_2(\mathcal{H}_{\mathfrak{p}};\mathbb{Z})$-summand in (IV.1.4).

Part 8: Fix $k \in \{0,\ldots,\mathrm{G}\}$. Given $T > 1$ and a metric from Met_T on Y_k, the next lemma uses w_T to denote the associated harmonic 2-form on Y_k whose de Rham cohomology class is that of $c_1(\det(\mathbb{S}))$.

Lemma 9.1 *There exists $\kappa > 1$ with the following significance: Fix a metric from Y_k's version of Met_T so as to define w_T. Let $\|p\|_2$ denote the metric L^2-norm of p, and let w be the closed 2-form from (3.5). Then the L^2-norm of w_T is at most $\kappa(1+\|p\|_2)$ and the C^1-norm of $w_T - w$ on $\mathcal{Y}_0$ and on the $r > \rho_* + \frac{1}{2}\varepsilon$ part of $\mathcal{N}_\varepsilon$ is at most $\kappa T^{-1/2}$.*

Proof. The proof has four steps.

Step 1: The L^2-norm of w_T as defined by the metric from Met_T on Y_k is greater than c_0^{-1} because the integral of w_T over $\mathcal{H}_0$ must be greater than c_0^{-1} so as to have integral 2 on each cross-sectional 2-sphere. As explained directly, the L^2-norm of w_T is also less than $c_0(1 + \|p\|_2)$. The proof that this is so uses the fact that a given harmonic form minimizes the L^2-norm amongst all closed forms in its de Rham cohomology class. To obtain such a form, reintroduce the coordinates (t,z) for U_γ and let B denote a smooth function with compact support centered on the origin in $\mathbb{C}$ and with integral 2. Choose a T-independent version of B so that its incarnation as a function on U_γ has support in $U_\gamma \cap \mathcal{H}_0$. With B chosen, set p_0 to denote $\frac{i}{2}\mathrm{B}\, dz \wedge d\bar{z}$. This is a closed, compactly supported 2-form in $\mathcal{Y}_0$ whose de Rham cohomology class when viewed in either $H^2(Y_k;\mathbb{Z})$ has pairing zero with all but the $H_2(\mathcal{H}_0;\mathbb{Z})$-summand in the Y_k version of (IV.1.4). By construction, the de Rham cohomology class of $p_{\mathbb{S}} = p_0 + p$ is that of $c_1(\det(\mathbb{S}))$. The metric L^2-norm of $p_{\mathbb{S}}$ is less than $c_0\,(1 + \|p\|_2)$.

Step 2: Use σ to denote the function on $\gamma^{(z_0)}$'s component of $Y_k - S$ that equals 1 on $\gamma^{(z_0)}$'s component of $Y_k - \mathcal{N}_\varepsilon$ and is given near S by the function on the $r \geq \rho_*$ part of $\mathbb{R}^3$ by the radial function $r \mapsto \chi(2 - 128\varepsilon^{-1}(r-\rho_*))$. The function σ is equal to 1 where $r > \rho_* + \frac{1}{64}\varepsilon$ and it is equal to 0 where $r < \rho_* + \frac{1}{128}\varepsilon$.

Use e_T to denote the Φ_T^{-1}-pull-back to $S^1 \times S^2$ of the 2-form σw_T. This 2-form is supported on the complement in $S^1 \times S^2$ of the $r < \frac{1}{T}(\rho_* + \frac{1}{128}\varepsilon)$ part of the distinguished coordinate ball. It follows from what is said in Step 1 that the L^2-norm of e_T is bounded from below by c_0^{-1} and bounded from above by c_0.

Use $*$ to denote the $\mathfrak{g}_*$-Hodge dual on $S^1 \times S^2$. Note that de_T and $d * e_T$ are equal to zero on the complement of the $r \leq \frac{1}{T}(\rho_* + \frac{1}{64}\varepsilon)$ part of the distinguished coordinate chart. Meanwhile, the norms of both are bounded by $c_0 T|(\Phi_T^{-1}) * w_T|_{\mathfrak{g}_*}$ on this same ball. This observation, the fact that the $\mathfrak{g}_T$-metric is the Φ_T-pullback of $\mathfrak{g}_*$ and the fact that the $\mathfrak{g}_*$-volume of the $r \leq \frac{1}{T}(\rho_* + \frac{1}{64}\varepsilon)$ coordinate ball is bounded by $c_0 T^{-3}$ implies that the L^1-norm of both de_T and $d * e_T$ is bounded by $c_0(1 + \|p\|_2)T^{-1/2}$.

Step 3: The 2-form w appears in the $r \geq \rho_*$ part of the $\mathbb{R}^3$ incarnation of $\mathcal{N}_\varepsilon$ as $\kappa(\rho)\rho d\rho \wedge d\phi$. The latter form extends smoothly to the $r \leq \rho_*$ part of $\mathbb{R}^3$ as a $\mathfrak{g}_*$-harmonic 2-form. It follows as a consequence that w's restriction to $\mathcal{Y}_0$ and to the $r \geq \rho_* + \frac{1}{32}\varepsilon$ part of $\mathcal{N}_\varepsilon$ is the pull-back by all Φ_T of the $\mathfrak{g}_*$-harmonic 2-form on $S^1 \times S^2$ whose de Rham class has pairing equal to 2 with the generator of $H_2(S^1 \times S^2; \mathbb{Z})$. This corresponding form on $S^1 \times S^2$ is $\frac{1}{2\pi} \sin\theta d\theta \wedge d\phi$ and also denoted by w.

Step 4: Introduce the operator $\mathfrak{D}_* = *d + d*$ on $S^1 \times S^2$ and use it to write the 2-form e_T as $(1 + \mathfrak{z}_T)w + u_T$ with $\mathfrak{z}_T$ denoting a constant with norm bounded by $c_0 T^{-3/2}$ and with u_T denoting a 2-form which is L^2-orthogonal to w and such that $\mathfrak{D}u_T = \mathfrak{D}e_T$. As the Green's function kernel for $\mathfrak{D}$ is smooth on the complement of the diagonal in $\times_2(S^1 \times S^2)$, the fact that $\mathfrak{D}e_T$ has support where $r < \frac{1}{T}(\rho_* + \frac{1}{64}\varepsilon)$ and the $c_0(1 + \|p\|_2)T^{-1/2}$ bound on its L^1-norm implies that $|u_T| + |\nabla u_T| \leq c_0(1 + \|p\|_2)T^{-1/2}$ on $\mathcal{Y}_0$ and also on the $r > \rho_* + \frac{1}{2}\varepsilon$ part of $\mathcal{N}_\varepsilon$. $\square$

9.2 Proof of Proposition 3.8

The three parts of this subsection prove the assertion made by Proposition 3.8.

Part 1: Let Y_Z denote a given compact, oriented 3-manifold and let z denote a non-zero class in $H^2(Y_Z; \mathbb{Z})/\operatorname{tors}$. Hodge theory associates to each metric on Y_Z a harmonic 2-form whose de Rham cohomology class is z. Of specific interest in what follows are metrics whose associated harmonic 2-form has transverse zeros. There is a residual set of metrics on Y_Z with this property, see for example [Ho] for a proof.

Fix $k \in \{0, \ldots, \mathrm{G}\}$. Let $\mathfrak{g}_{\mathcal{N}}$ denote a metric in the Y_k version of $\mathrm{Met}^{\mathcal{N}}$. Fix $T > 1$ and use $\mathfrak{g}_{\mathcal{N}}$ to define a metric in Met_T, this denoted by $\mathfrak{g}_1$. Let w_1 denote the associated harmonic 2-form with de Rham cohomology class $c_1(\det(\mathbb{S}))$. If w_1 has degenerate zeros, fix a second metric, $\mathfrak{g}_2$, on Y_k with the following properties: Let w_2 denote the

corresponding $\mathfrak{g}_2$ harmonic 2-form. Then w_2 has non-degenerate zeros, and the $\mathfrak{g}_1$ norms of $w_2 - w_1$ and $\mathfrak{g}_2 - \mathfrak{g}_1$, and those of their $\mathfrak{g}_1$-covariant derivatives to order 100 are less than T^{-1}. If w_1 has nondegenerate zeros, take $\mathfrak{g}_2 = \mathfrak{g}_1$.

Part 2: Write w_2 on $\mathcal{Y}_0$ and on the $r > \rho_* + \frac{1}{2}\varepsilon$ part of $\mathcal{N}_\varepsilon$ as $w + u_2$. By Lemma 9.1, the 2-form is such that $|u_2| \leq c_0 T^{-1/2}$. This 2-form is also exact; but more to the point, u_2 can be written as dz_2 where z_2 is a 1-form with $|z_2| < c_0 T^{-1/2}$ on the $r \geq \rho_* + \frac{5}{8}\varepsilon$ part of $\mathcal{N}_\varepsilon$. Hold on to z_2 for the moment. Let $\sigma_\perp$ denote the function of r on $\mathcal{N}_\varepsilon$ given by $\sigma_\perp = \chi(8\varepsilon^{-1}(r - \rho_*) - 5)$. This function is equal to 1 where $r < \rho_* + \frac{5}{8}\varepsilon$ and it is equal to 0 where $r > \rho_* + \frac{3}{4}\varepsilon$. Use w_3 to denote the closed 2-form on Y_* that is given by w_2 on $\mathcal{Y}_M$, given by w on $\mathcal{Y}_0$ and given by $w + d(\sigma_\perp z_2)$ on $\mathcal{N}_\varepsilon$. The 2-form w_3 has the same de Rham class as w_2, the same zero locus as it agrees with w_2 where both are zero, and $|w_2 - w_3| \leq c_0 T^{-1/2}$.

Use $\upsilon_\diamond$ to denote the $\mathfrak{g}_*$-Hodge dual on $S^1 \times S^2$ of the 2-form $w = \sin\theta\, d\theta \wedge d\phi$. Write the $\mathfrak{g}_2$-Hodge star of w_2 as $\upsilon_\diamond + q_2$ on $\mathcal{Y}_0$ and on the r $> \rho_* + \frac{1}{2}\varepsilon$ part of $\mathcal{N}_\varepsilon$. As both the $\mathfrak{g}_2$-Hodge star of w_2 and $\upsilon_\diamond$ are exact on $\mathcal{N}_\varepsilon$, it follows that $q_2 = d o_2$ on $\mathcal{N}_\varepsilon$. Moreover, such a function o_2 can be found with $|o_2| \leq c_0 T^{-1/2}$ on the $r > \rho_* + \frac{1}{2}\varepsilon$ part of $\mathcal{N}_\varepsilon$. This is so because $|w - w_2| < c_0 T^{-1}$ and $|\mathfrak{g}_2 - \mathfrak{g}_*| < c_0 T^{-1}$ on this part of $\mathcal{N}_\varepsilon$. Fix a version of o_2 that obeys this bound. Let υ_3 denote the closed 1-form on Y_* given by $\upsilon_\diamond$ on $\mathcal{Y}_0$, by the $\mathfrak{g}_2$-Hodge star of w_2 on $\mathcal{Y}_M$ and given on $\mathcal{N}_\varepsilon$ by $\upsilon_\diamond + d(\sigma_\perp o_2)$. This closed 1-form is such that $w_3 \wedge \upsilon_3 \geq 0$ when $T > c_0$ with equality only at the zeros of w_3.

With $T > c_0$ chosen, the upcoming Lemma 9.2 uses what was just said about w_3 and υ_3 as input to supply a metric on Y_* with the properties in the list that follows. This new metric is denoted by $\mathfrak{g}_{3T}$. The $\mathfrak{g}_{3T}$-Hodge star sends w_3 to υ_3; thus w_3 is $\mathfrak{g}_{3T}$-harmonic. The metric $\mathfrak{g}_{3T}$ on $\mathcal{Y}_0$ and on the $r > \rho_* + \frac{3}{4}\varepsilon$ part of $\mathcal{N}_\varepsilon$ is the metric $\mathfrak{g}_*$. The metric $\mathfrak{g}_{3T}$ on the $r \in [\rho_* + \frac{1}{2}\varepsilon, \rho_* + \frac{3}{4}\varepsilon]$ part of $\mathcal{N}_\varepsilon$ can be written as $\mathfrak{g}_2 + \mathfrak{h}$ with $\mathfrak{h}$ and its $\mathfrak{g}_2$-covariant derivaties to order 20 having $\mathfrak{g}_2$-norm less than $c_0 T^{-1}$. Finally, the metrics $\mathfrak{g}_{3T}$ and $\mathfrak{g}_2$ are identical except on the rest of Y.

Any sufficiently large T version of the metric $\mathfrak{g}_{3T}$ meets the requirements of Proposition 3.8's space *Met*. Conversely, each metric in *Met* is a sufficiently large T version of a metric $\mathfrak{g}_{3T}$ that is constructed as described above from some metric in $\mathrm{Met}^{\mathcal{N}}$. The lower bound on T depends on various properties of the chosen $\mathrm{Met}^{\mathcal{N}}$ metric, these being an upper bound on the norm of the metric's Riemann curvature, the metric volume of $\mathcal{Y}_M$, and a lower bound on the metric's injectivity radius.

Part 3: The existence of the metric $\mathfrak{g}_{3T}$ follows from the first lemma below.

Lemma 9.2 *Let Y_Z denote an oriented 3-manifold and let $\mathfrak{g}$ denote a given Riemannian metric on Y_Z. Use $*$ in what follows to denote the Hodge star defined by $\mathfrak{g}$. Suppose that U and V are open sets in Y_Z with the closure of V being a compact subset of U. Let ω and υ denote respectively a 2-form and a 1-form on Y_Z such that $\omega \wedge \upsilon > 0$ on U and such that $*\omega = \upsilon$ on $Y_Z - V$.*

- *There are smooth metrics on Y_Z which equal $\mathfrak{g}$ on $Y_Z - U$ and have Hodge star sending ω to υ. Moreover, there exists metric of this sort whose volume 3-form is the same as the $\mathfrak{g}$-volume 3-form.*
- *Fix $k \in \{0, 1, \ldots\}$ and $\mathrm{D} > 1$. There exists $\kappa > 1$ with the following significance: Suppose that the C^k-norms on U of ω, υ and the Riemann curvature tensor of $\mathfrak{g}$ are less than D. Then Y_Z has a metric that obeys the conclusions of the first bullet and differs from $\mathfrak{g}$ by a tensor whose $\mathfrak{g}$-norm and those of its $\mathfrak{g}$-covariant derivatives to order k are bounded by κ times the C^k-norm of $*\omega - \upsilon$.*

Lemma 9.2 has a generalization that holds for 1-parameter families of data sets. This parametrized version is given below but used in the next subsection.

Lemma 9.3 *Let $\{(\mathfrak{g}_\tau, \omega_\tau, \upsilon_\tau)\}_{\tau\in[0,1]}$ denote a smoothly parametrized family of metrics, 2-forms and 1-forms on Y_Z with $\omega_\tau \wedge \upsilon_\tau > 0$ on U and such that the $\mathfrak{g}_\tau$-Hodge dual of ω_τ is υ_τ on V. There is a corresponding smooth, 1-parameter family of metrics such that each $\tau \in [0,1]$ member obeys the conclusion of first bullet of Lemma 9.2. Moreover, this new family of metrics can be chosen to obey the properties listed below.*

- *Let $I \subset [0,1]$ denote an open neighborhood of one or both of the end points. Suppose that the conclusions of the first bullet of Lemma 9.2 hold for $(\mathfrak{g}_\tau, \omega_\tau, \upsilon_\tau)$ when $\tau \in I$. There is a neighborhood $I' \subset I$ of the endpoints such each $\tau \in I'$ member of the new family is the corresponding $\mathfrak{g}_\tau$.*
- *Given a non-negative integer k and $\mathrm{D} \geq 1$, there exists $\kappa > 1$ with the following significance: Suppose that the conditions of the second bullet of Lemma 9.2 are satisfied for each $\tau \in [0,1]$ and that the C^k-norms of the τ-derivatives to order k of $\{(g_\tau, \omega_\tau, \upsilon_\tau)\}_{\tau\in[0,1]}$ are also bounded by D. There is a 1-parameter family of metrics that obeys the preceding bullet and the first and second bullets of Lemma 9.2. In addition, each $\tau \in [0,1]$ member of the family differs from the corresponding metric $\mathfrak{g}_\tau$ by a tensor whose τ-derivatives to order k have C^k-norm bounded by κ times the C^k-norm of the sum of the τ-derivatives to order k of the difference between υ_τ and the $\mathfrak{g}_\tau$-Hodge star of ω_τ.*

Proof of Lemmas 9.2 and 9.3. Let Ω denote $\mathfrak{g}$'s volume 3-form. Write $\omega \wedge \upsilon$ as $\mathfrak{q}\Omega$ with $\mathfrak{q}$ being a non-negative function on U. Let v denote the vector field on U that is annihilated by ω and has pairing $\mathfrak{q}$ with υ. Let $\mathrm{Ker}(\upsilon) \subset TU$ denote the 2-plane bundle that is annihilated by υ. The 2-form ω is symplectic on $\mathrm{Ker}(\upsilon)$ and so orients $\mathrm{Ker}(\upsilon)$. Choose an ω-compatible almost complex structure on $\mathrm{Ker}(\upsilon)$, denoted by J below. Note in this regard that there are no obstructions to finding such an almost complex structure. This is so because the space of almost complex structures that are compatible with a constant symplectic form on $\mathbb{R}^2$ is contractible. The construction just given yields a new metric with volume 3-form Ω.

With J chosen, a metric on U is defined as follows: The vector field v has norm $\mathfrak{q}^{1/2}$ and is orthogonal to $\mathrm{Ker}(\upsilon)$. The inner product between vectors v and v' in a given fiber of $\mathrm{Ker}(\upsilon)$ is $\mathfrak{q}^{-1/2}\omega(v, Jv')$. A metric of this sort has $*\omega = \upsilon$ and is such that both ω and υ have norm $\mathfrak{q}^{1/2}$. Moreover, any metric with these two properties is of the form just described. In particular, any two differ only with respect to the choice of the almost complex structure on the $\mathrm{Ker}(\upsilon)$.

Let J_1 denote a chosen, ω-compatible almost complex structure on $\mathrm{Ker}(\upsilon)|_U$ and let $\mathfrak{g}_1$ denote the corresponding metric. The metric $\mathfrak{g}$ on $U - V$ is by necessity of the sort just described, thus it differs from $\mathfrak{g}_1$ only on $\mathrm{Ker}(\upsilon)$. In particular, the metric $\mathfrak{g}$ on $\mathrm{Ker}(\upsilon)$ is given by $\mathfrak{q}^{-1/2}\omega(v, J_{\mathfrak{g}}v')$ with $J_{\mathfrak{g}}$ being an ω-compatible almost complex structure on $\mathrm{Ker}(\upsilon)|_{U-V}$. As noted above, if point $p \in U$, then the space of $\omega|_p$-compatible almost complex structures on $\mathrm{Ker}(\upsilon)|_p$ is contractible. This understood, there are no obstructions to choosing an ω-compatible almost complex structure on $\mathrm{Ker}(\upsilon)|_U$ that agrees with $J_{\mathfrak{g}}$ near $Y_Z - U$ and agrees with J_1 on V. Let J_2 denote an almost complex structure of this sort. The metric defined as instructed above by J_2 has the properties that are asserted by the first bullet of Lemma 9.2.

The assertions of the second bullet of Lemma 9.2 and those of Lemma 9.3 are proved by taking care with the choice of J_2 and its $\tau \in [0, 1]$ counterparts. As the details are straightforward and rather tedious, they are omitted. □

9.3 Met_T metrics on cobordisms

Lemma 9.1 has an analog given below that concerns self-dual forms on cobordisms. The cobordism manifold is denoted below by X and it is assumed to be of the sort that is described in Section 3.3 with its constant s slices where $s < -1$ and $s > 1$ given as follows: Either one is Y and the other is Y_{G}; or one is some $k \in \{1, \ldots, \mathrm{G}\}$ version of Y_k and the other is $Y_{k-1} \sqcup (S^1 \times S^2)$, or one is Y_0 and the other is $M \sqcup S^1 \times S^2$.

The case when both are Y or both some $k \in \{1, \dots, \mathrm{G}\}$ version of Y_k is also allowed, but only the case where both are Y_{G} are needed in what is to come. The topology of X is further constrained by the requirement that s have 1 critical point when it is not diffeomorphic to a product with $\mathbb{R}$. If one of these slices is Y and the other Y_{G}, or if both are Y or both Y_k for $k \in \{1, \dots, \mathrm{G}\}$, then s has no critical points and the cobordism manifold X is $\mathbb{R} \times Y$ or $\mathbb{R} \times Y_k$ as the case may be, with the projection to $\mathbb{R}$ given by the function s.

One more constraint on X is needed. By way of background, what is said in Part 4 of Section 9.1 identifies $\mathcal{Y}_0 \cup \mathcal{N}_\varepsilon$ as a subset of Y and Y_k, and also $S^1 \times S^2$. This extra constraints uses $\mathcal{Y}_{0\varepsilon}$ to denote the union of $\mathcal{Y}_0$ and the $r > \rho_* + \frac{1}{128}\varepsilon$ part of $\mathcal{N}_\varepsilon$. Here is the extra constraint:

(9.10)

There is a distinguished embedding of $\mathbb{R} \times \mathcal{Y}_{0\varepsilon}$ into X with the following property:
The respective $s < 0$ and $s > 0$ slices of the image of this embedding,
when written using the diffeomorphisms from the second and third bullets of (2.7),
appear as the incarnation of $\mathcal{Y}_{0\varepsilon}$ in either Y, Y_k, or $S^1 \times S^2$ as the case may be.

The metric for X is assumed to obey a constraint that requires membership in an analog for X of the various $T > 1$ versions of the space Met_T. The definition of this X version of Met_T requires the a priori selection of metrics $\mathfrak{g}_-$ and $\mathfrak{g}_+$ from the respective Y_- and Y_+ versions of Met_T with it understood that Met_T in the case of $M \sqcup (S^1 \times S^2)$ is the space consisting of the metric $\mathfrak{g}_*$ on $S^1 \times S^2$ and a metric on M of the following sort: If $c_1(\det(\mathbb{S}))$ is torsion on M, then any metric on M is allowed. If this class is not torsion, then the metric's associated harmonic 2-form with de Rham coholomogy class $c_1(\det(\mathbb{S}|_M))$ has non-degenerate zeros. Meanwhile, Met_T for any given $k \in \{1, \dots, \mathrm{G}\}$ version of $Y_k \sqcup (S^1 \times S^2)$ consists of a Met_T metric for Y_{k-1} and any metric for $S^1 \times S^2$. Reintroduce from Part 5 of Section 9.1 the metric $\mathfrak{g}_T$ on $\mathcal{Y}_0 \cup \mathcal{N}_\varepsilon$. Of immediate interest in what follows is $\mathfrak{g}_T$'s restriction to $\mathcal{Y}_{0\varepsilon}$. By way of a reminder, $\mathfrak{g}_T$ on $\mathcal{Y}_{0\varepsilon}$ is the metric $\mathfrak{g}_*$ on $\mathcal{Y}_0$ and it is the metric in (9.7) on the $r > \rho_* + \frac{1}{128}\varepsilon$ part of $\mathcal{N}_\varepsilon$.

The analog of Met_T for X consists of the space of metrics with the following three

properties:

$$(9.11)\quad \begin{cases} \bullet\ \text{The metric obeys the } L = 100 \text{ version of (2.8).} \\ \bullet\ \text{The metric pulls back via the embedding in (9.10) as the metric } ds^2 + \mathfrak{g}_T. \\ \bullet\ \text{The metric pulls back from the } s \leq -104 \text{ part of } X \text{ via the embedding} \\ \quad \text{in the second bullet of (3.8) as } ds^2 + \mathfrak{g}_-, \text{ and it pulls back from the} \\ \quad s > 104 \text{ part of } X \text{ via the embedding from the third bullet of (2.7) as} \\ \quad ds^2 + \mathfrak{g}_+. \end{cases}$$

This analog for X of Met_T is denoted in what follows by Met_T also, its dependence on $\mathfrak{g}_-$ and $\mathfrak{g}_+$ being implicit.

Lemma 9.4 given momentarily supplies the promised analog to Lemma 9.1. To set the notation, suppose that a metric on X has been specified and that p_X is a differential form on X. The lemma uses $\langle p_X \rangle_2$ to denote the L^2-norm of p_X over the $|s| < 104$ part of X. Lemma 9.4 uses w_- and w_+ to denote the respective $\mathfrak{g}_-$ and $\mathfrak{g}_+$ harmonic 2-forms with de-Rham cohomology class that of $c_1(\det(\mathbb{S}))$; and it uses the embeddings from the second and third bullets of (2.7) to view w_- and w_+ as 2-forms on the $s \leq -1$ and $s > 1$ parts of X.

Lemma 9.4 *Let X denote a cobordism manifold of the sort described above. Given metrics $\mathfrak{g}_-$ and $\mathfrak{g}_+$ in the respective Y_- and Y_+ versions of Met_T, there exists $\kappa > 1$ with the following significance: Fix $T > 1$, and fix a Riemannian metric on X from the corresponding set Met_T. There is a self-dual, harmonic 2-form on X whose pull-back to the constant s-slices of X converges as $s \to -\infty$ to w_- and as $s \to \infty$ to w_+. Let p_X denote a closed 2-form on X that equals w_- where $s < -102$, that equals w_+ where $s > 102$, and with de Rham cohomology class that of $c_1(\det(\mathbb{S}))$.*

- *The L^2-norm of this harmonic self-dual 2-form on the s-inverse image of any length 1 interval in $\mathbb{R}$ is bounded by $\kappa\langle p_X \rangle_2$.*
- *The pull-back of this harmonic self-dual 2-form to the constant $s > 1$ and $s < -1$ slices differs in the C^1-topology from w_- and w_+ by at most $\kappa\langle p_X \rangle_2\, e^{-|s|/z}$ with $z \geq 1$ depending on the corresponding limit metric.*
- *The pull-back of this harmonic self-dual 2-form to $\mathbb{R} \times \mathcal{Y}_{0\varepsilon}$ via the embedding from (9.10) differs from $ds \wedge \upsilon_\diamond + w$ by a 2-form whose C^1-norm on $\mathbb{R} \times \mathcal{Y}_0$ and on the $r > \rho_* + \frac{1}{2}\varepsilon$ part of $\mathbb{R} \times \mathcal{N}_\varepsilon$ is less than $\kappa\langle p_X \rangle_2 T^{-1/2}$.*

Proof. The existence of a closed, self-dual harmonic 2-form with the desired $s \to -\infty$ and $s \to \infty$ limits follows from the index theorem in [APS]. This 2-form is denoted in

what follows by ω. Given the first bullet, then the assertion in the second bullet follows from the eigenfunction expansion that is depicted below in (9.13). As explained next, the third bullet also follows from the second bullet.

To prove the third bullet, fix $s_0 \subset \mathbb{R}$ and introduce σ_0 to denote the function on $\mathbb{R}$ given by the rule $s \mapsto \chi(|s - s_0| - 1)$. This function equals 1 where $|s - s_0|$ is less than 1 and it equals zero where $|s - s_0|$ is greater than 2. Let σ denote the function from Step 2 of the proof of Lemma 9.1 and let Φ_T denote the embedding from Part 5 of Section 9.1. View the Φ_T^{-1} pull-back of $\sigma_0\sigma\omega$ as a 2-form on $\mathbb{R} \times (S^1 \times S^2)$ with support where $|s - s_0| < 2$. The assumed L^2-bound for ω with a Green's function argument much like that used in Step 4 of the proof of Lemma 9.1 can be used to derive the pointwise bound that is asserted by Lemma 9.4. The derivation differs little from that in Step 4 of the proof of Lemma 9.1 save for the fact that the Green's function in question is that for the elliptic operator

$$(9.12)\quad \mathcal{D}\colon C^\infty(\mathbb{R} \times (S^1 \times S^2); \Lambda^+ \oplus \mathbb{R}) \to C^\infty(\mathbb{R} \times (S^1 \times S^2); T^*(\mathbb{R} \times (S^1 \times S^2))$$

given by the formula $\mathcal{D} = *_X d_X + d_X$ where d_X denotes the 4-dimensional exterior derivative $ds \wedge \frac{\partial}{\partial s}(\cdot) + d$ and where $*_X$ denotes the Hodge star for the metric $ds^2 + \mathfrak{g}_*$.

The lemma's first bullet is proved in the four steps that follow.

Step 1: Let ω denote the relevant closed, self-dual harmonic form. Fix an integer $n \in \{106, 107, \ldots\}$ and introduce by way of notation $I_n \subset \mathbb{R}$ to denote a closed interval of length $2n$ whose endpoints have distance 106 or more from the origin. Let C denote the space of closed two forms on the domain $s^{-1}(I_n)$ that agree with ω on some neighborhood of the s-inverse images of the boundary points of I_n. The 2-form ω is the minimizer in C of the functional that is defined by the rule $\mathfrak{w} \mapsto \int_{s^{-1}(I_n)} |\mathfrak{w}^+|^2$.

Step 2: Use the embedding from the second bullet of (2.7) to write the $s \geq 100$ part of X as $[100, \infty) \times Y_+$ and likewise write the $s < -100$ part of X as $(-\infty, -100] \times Y_-$. Let Y_* for the moment denote either Y_+ or Y_-. Let $*$ denote either the $\mathfrak{g}_-$- or $\mathfrak{g}_+$-version of the Hodge star on Y_*. The corresponding operator $d*$ defines an unbounded, self-adjoint operator on the space of closed 2-forms on Y_*. Let Ξ^- denote an L^2-orthonormal basis of eigenvectors of $d*$ on the space of closed 2-forms with negative eigenvalue and let Ξ^+ denote an L^2-orthonormal basis of eigenvectors of $d*$ with positive eigenvalue. The eigenvalue of $d*$ on a given eigenvector, a, is denoted by λ_{a}.

The 2-form ω on $(-\infty,-1]\times Y_-$ and on $[1,\infty)\times Y_+$ can be written as

(9.13)
$$\begin{cases}\omega = ds\wedge *w_- + w_- + \sum_{\mathrm{a}\in\Xi^+} z_{\mathrm{a}}\, e^{\lambda_{\mathrm{a}}(s+1)}(ds\wedge *\mathrm{a}+\mathrm{a}) & \text{where } s\le -104.\\ \omega = ds\wedge *w_+ + w_+ + \sum_{\mathrm{a}\in\Xi^-} z_{\mathrm{a}}\, e^{\lambda_{\mathrm{a}}(s-1)}(ds\wedge *\mathrm{a}+\mathrm{a}) & \text{where } s\ge 104.\end{cases}$$

What is denoted by $z_{(\cdot)}$ in (9.13) is a real number. Keep in mind for what follows that any given version of $e^{\lambda_{\mathrm{a}}s}(ds\wedge *\mathrm{a}+\mathrm{a})$ is the exterior derivative on its domain of definition of the 1-form $q_{\mathrm{a}} = \lambda_{\mathrm{a}}^{-1}e^{\lambda_{\mathrm{a}}s} * \mathrm{a}$.

Step 3: Fix $m>1$. Let a denote an eigenvector in the Y_- version of Ξ^+. Introduce σ_{a} to denote the function on $\mathbb{R}$ given by the rule $s\mapsto \sigma_{\mathrm{a}}(s) = 1-\chi(-m^{-1}\lambda_{\mathrm{a}}(s+102)-1)$. This function equals 0 where $s>-102-m\lambda_{\mathrm{a}}^{-1}$ and it equals 1 where $s<-102-2m\lambda_{\mathrm{a}}^{-1}$. If a is in the Y_+ version of Ξ^-, then σ_{a} is given by the rule $s\mapsto\sigma_{\mathrm{a}}(s) = 1-\chi(-m^{-1}\lambda_{\mathrm{a}}(s-102)-1)$. This version of σ_{a} is 0 where $s<102+m|\lambda_{\mathrm{a}}|^{-1}$ and it is 1 where $|s|>102+2m|\lambda_{\mathrm{a}}|^{-1}$. Meanwhile, use χ_* to denote the function $\chi(102-|s|)$. This function is 1 where $|s|>102$ and 0 where $|s|<101$.

Use p_X and these functions to define the 2-form $\mathfrak{w}$ on X by the rule

(9.14)
$$\mathfrak{w} = \chi_* ds\wedge *p_X + p_X + \sum_{\mathrm{a}\in\Xi^+} z_{\mathrm{a}}\, d(\sigma_{\mathrm{a}}\lambda_{\mathrm{a}}^{-1}e^{\lambda_{\mathrm{a}}(s+102)} * \mathrm{a}) + \sum_{\mathrm{a}\in\Xi^-} z_{\mathrm{a}}\, d(\sigma_{\mathrm{a}}\lambda_{\mathrm{a}}^{-1}e^{\lambda_{\mathrm{a}}(s-102)} * \mathrm{a}).$$

This is a closed 2-form whose de Rham cohomology class is the same as ω. Let E denote the smallest of the numbers from the set $\{\lambda_{\mathrm{a}} \mid \mathrm{a}\in\Xi^+\}\cup\{|\lambda_{\mathrm{a}}| \mid \mathrm{a}\in\Xi^-\}$ with it understood that Ξ^+ refers to the Y_- version and Ξ^- refers to the Y_+ version. The 2-form $\mathfrak{w}$ equals ω where $|s|\ge 1+2m\,\mathrm{E}^{-1}$.

Step 4: The square of the L^2-norm of $\mathfrak{w}^+$ over the $|s|\le 102+2m\mathrm{E}^{-1}$ part of X is no greater than

(9.15)
$$\int_{s^{-1}([-102,102])}|p_X|^2 + c_0 m^{-2}e^{-2m}\sum_{\mathrm{a}\in\Xi^+\cup\Xi^-}|\lambda_{\mathrm{a}}|^{-1}|z_{\mathrm{a}}|^2 + 4m\mathrm{E}^{-1}(\|w_-\|_2^2+\|w_+\|_2^2) + \sum_{\mathrm{a}\in\Xi^+\cup\Xi^-}|\lambda_{\mathrm{a}}|^{-1}|z_{\mathrm{a}}|^2(e^{-2m}-e^{-4|\lambda_{\mathrm{a}}|m/\mathrm{E}}).$$

Meanwhile, the integral of ω over this same part of X is equal to

(9.16)

$$\int_{s^{-1}([-102,102])} |\omega|^2 + \sum_{\mathrm{a}\in\Xi^+\cup\Xi^-} |\lambda_{\mathrm{a}}|^{-1}|z_{\mathrm{a}}|^2(1-e^{-2m}) + 4m\mathrm{E}^{-1}(\|w_-\|_2^2 + \|w_+\|_2^2)$$
$$+ \sum_{\mathrm{a}\in\Xi^+\cup\Xi^-} |\lambda_{\mathrm{a}}|^{-1}|z_{\mathrm{a}}|^2(e^{-2m} - e^{-4|\lambda_{\mathrm{a}}|m/\mathrm{E}}).$$

As noted in Step 1, the expression in (9.16) can not be greater than what is written in (9.15). This being the case, the $m > c_0$ versions of (9.15) and (9.16) imply the bound

$$\int_{s^{-1}([-102,102])} |\omega|^2 + \sum_{\mathrm{a}\in\Xi^+\cup\Xi^-} |\lambda_{\mathrm{a}}|^{-1}|z_{\mathrm{a}}|^2 \leq c_0(1 + \langle p_X\rangle_2^2). \tag{9.17}$$

This last bound has the following corollary: Let $I \subset \mathbb{R}$ denote any interval of length 1. Then $\int_{s^{-1}(I)} |\omega|^2 \leq c_0\,(1 + \langle p_X\rangle_2^2)$. □

9.4 Proof of Proposition 3.10

To explain the first bullet, identify a neighborhood of the critical point of the function s with a ball about the origin in $\mathbb{R}^4$ using coordinates (y_1, y_2, y_3, y_4) and write s in terms of these coordinates as $s = y_4^2 - y_1^2 - y_2^2 - y_3^2$ when the constant, $s < -1$ slices of X are Y_0 and the constant, $s > 1$ slices are $M \sqcup (S^1 \times S^2)$. With the ends reversed, the function s appears as $s = -y_4^2 + y_1^2 + y_2^2 + y_3^2$. The embeddings given in the second and third bullets of (2.7) are defined using a pseudogradient vector field for s. This pseudogradient vector field in the $Y_- = Y_0$ and $Y_+ = M \sqcup (S^1 \times S^2)$ case can be chosen so as to have the following properties: The inverse image of the descending 3-ball from the critical point via the embedding given by the second bullet of (2.7) appears as the locus $(-\infty, 0) \times S$ with S being the 2-sphere that is described in Part 4 of Section 9.1. Meanwhile, the inverse image via the embedding given by the third bullet of (2.7) of one of the ascending arc from this critical point intersects the $(0, \infty) \times (S^1 \times S^2)$ component of $(0, \infty) \times (M \sqcup (S^1 \times S^2))$ as the locus $(0, \infty) \times p_*$ with $p_* \in S^1 \times S^2$ being the $r = 0$ point in the ball that is described in the third paragraph of Part 4 in Section 9.1. The other ascending arc intersects the $(0, \infty) \times M$ component as the $r' = 0$ point in the ball that is described in the fourth paragraph of Section 9.1. There is a completely analogous picture of X when Y_0 is the constant $s > 0$ slice of X and $S^1 \times S^2$ is the constant $s < 0$ slice.

What is said above about the descending and ascending submanifolds from the critical point has the following consequence: The pseudogradient vector field that defines the embeddings from the second and third bullets of (2.7) can be chosen so that (3.9) are obeyed and likewise the condition in (9.10). These properties are assumed in what follows. The fact that S carries no homology implies that the fourth bullet of (2.7) holds for X.

Parts 1-10 of this subsection construct large L versions of the form w_X and the metric that are used in Part 11 to satisfy the requirements of the second bullet of Proposition 3.10. These constructions require the choice of parameters $T \gg 1$, $L_0 \gg 1$ and $L_1 > L_0 + 1$. Granted large choices, Parts 1-10 construct a closed 2-form denoted by ω_{T*} and a metric denoted by $\mathfrak{m}_{T*}$ that makes ω_{T*} self-dual. Any $L > L_1 + 20$ version of ω_{T*} can serve for Proposition 3.10's desired 2-form w_X and the corresponding version of $\mathfrak{m}_{T*}$ can serve for the desired metric.

Proposition 3.10 requires as input a metric on $M \sqcup (S^1 \times S^2)$ and asserts that such a metric determines a certain subset of the set *Met* on Y_0. To say more about this subset, recall from Part 2 of Section 9.2 that each metric in *Met* is determined in part by a metric from Section 9.1's set $\mathrm{Met}^{\mathcal{N}}$ and a large choice for a number denoted by T. A metric of this sort was denoted by $\mathfrak{g}_{3T}$ in Section 9.2. As noted at the end of Part 2 of Section 9.2, a lower bound on T is determined by certain properties of the metric from $\mathrm{Met}^{\mathcal{N}}$. A metric of this sort is in Proposition 3.10's subset if and only if T is greater than a new lower bound that is determined by the afore-mentioned properties of the $\mathrm{Met}^{\mathcal{N}}$ metric. Suffice it to say for the purposes of the proof that this new lower bound is defined implicitly by the constructions in the subsequent eleven parts of this subsection.

The upcoming Parts 1-10 are written so as to simultaneously supply a metric and a closed, self-dual 2-form for Section 9.5's proof of Proposition 3.12 and Section 9.6's proof of Proposition 3.13. This is done by considering a cobordism space X as described in the previous section whose limit manifolds Y_- and Y_+ are as follows: Either one is Y_0 and the other is $M \sqcup (S^1 \times S^2)$; or one is some $k \in \{1, \ldots, \mathrm{G}\}$ version of Y_k and the other is $Y_{k-1} \sqcup (S^1 \times S^2)$, or both are Y_{G}. Although not needed for what follows, the constructions in Parts 1-10 can be done when both limit manifolds are Y or both are some $k \in \{1, \ldots, \mathrm{G}\}$ version of Y_k.

Part 1: When Y_- or Y_+ is not $M \sqcup (S^1 \times S^2)$, choose metrics, $\mathfrak{g}_{1-}$ and $\mathfrak{g}_{1+}$ in the respective Y_- and Y_+-versions of $\mathrm{Met}^{\mathcal{N}}$ as the case may be. In the case when one of Y_- or Y_+ is some $k \in \{1, \ldots, \mathrm{G}\}$ version of Y_k and the other is $Y_{k-1} \sqcup (S^1 \times S^2)$,

what is denoted $\mathrm{Met}^{\mathcal{N}}$ allows any metric for the $S^1 \times S^2$ component. Fix a $T \gg 1$; in particular so that Lemma 9.1 can be invoked for the metric in Met_T defined using $\mathfrak{g}_{1-}$ in the case of Y_- and $\mathfrak{g}_{1+}$ in the case of Y_+. Use $\mathfrak{g}_{1-}$ to choose a metric $\mathfrak{g}_2$ as directed in Part 2 of Section 9.2 on Y_-. Then set $\mathfrak{g}_- = \mathfrak{g}_2$. Meanwhile, use $\mathfrak{g}_2$ to construct a version of the metric $\mathfrak{g}_{3T}$ and denote it by $\mathfrak{g}_{-T}$. Do the same using $\mathfrak{g}_{1+}$; denote the chosen $\mathfrak{g}_2$ metric on Y_+ by $\mathfrak{g}_+$ and use $\mathfrak{g}_{+T}$ to denote the resulting $\mathfrak{g}_{3T}$ metric. If either of Y_- or Y_+ is $M \sqcup (S^1 \times S^2)$, take the metric of the sort described in Part 1 of Section 3.6 for M and the metric $\mathfrak{g}_*$ on $S^1 \times S^2$. Denote the resulting metric on $M \sqcup (S^1 \times S^2)$ as $\mathfrak{g}_-$ in the Y_- case and $\mathfrak{g}_+$ in the Y_+ case. With $T \geq 1$ chosen, this same metric is also denoted at times by $\mathfrak{g}_{-T}$ and $\mathfrak{g}_{+T}$ as the case may be.

By way of notation, the constant c_0 in what follows depends implicitly on the various properties of the metrics $\mathfrak{g}_{1-}$ and $\mathfrak{g}_{1+}$. In particular, c_0 depends on an upper bound for the norm of the metricâ^s curvature, upper and lower bounds on the metricâ^s volume and a lower bound on the injectivity radius.

Let $\mathfrak{m}$ denote a chosen metric in the $\mathfrak{g}_-$ and $\mathfrak{g}_+$ version of Met_T on X. Certain constraints on $\mathfrak{m}$ are imposed later in this subsection. Note that some of the latter impose constraints on $\mathfrak{g}_{1-}$ and $\mathfrak{g}_{1+}$.

Part 2: Use w_- and w_+ to denote the respective $\mathfrak{g}_-$ and $\mathfrak{g}_+$ harmonic 2-forms on Y_- and Y_+ with de Rham cohomology class that of $c_1(\det(\mathbb{S}))$. Fix for the moment a closed 2-form p_X on X as described in Lemma 9.4. Use ω to denote the self-dual 2-form on X given by Lemma 9.4 for the case when the metric on X is $\mathfrak{m}$. The distinguished embedding from (9.11) pulls ω back to $\mathbb{R} \times \mathcal{Y}_{0\varepsilon}$ as a 2-form that can be written as

$$\omega = ds \wedge \upsilon_\diamond + w + ds \wedge \frac{\partial}{\partial s} q + dq, \tag{9.18}$$

with q being an s-dependent 1-form on $\mathcal{Y}_{0\varepsilon}$. Lemma 9.4 says that the C^1-norms of $\frac{\partial}{\partial s} q$ and dq on $\mathbb{R} \times \mathcal{Y}_0$ and on the $r > \rho_* + \frac{1}{2}\varepsilon$ part of $\mathbb{R} \times \mathcal{N}_\varepsilon$ are less than $c_0 \langle p_X \rangle_2 T^{-1/2}$.

An s- and T-independent open cover of $\mathcal{Y}_{0\varepsilon}$ by balls of radius $c_0^{-1}\varepsilon$ can be used to write q on $\mathcal{Y}_0$ and on the $r > \rho_* + \frac{17}{32}\varepsilon$ part of $\mathcal{N}_\varepsilon$ as $q_0 + d\mathfrak{k}$ with q_0 obeying $|q_0| \leq c_0 \langle p_X \rangle_2 T^{-1/2}$ and $|\frac{\partial}{\partial s} q_0| \leq c_0 \langle p_X \rangle_2 T^{-1/2}$. Meanwhile, $\mathfrak{k}$ is a smooth function with $|d(\mathfrak{k})| \leq c_0 \langle p_X \rangle_2 T^{-1/2}$. Both q_0 and $\mathfrak{k}$ can be constructed so as to depend smoothly on s. It follows as a consequence of the bound $|d(\frac{\partial}{\partial s} \mathfrak{k})| \leq c_0 \langle p_X \rangle_2 T^{-1/2}$ that an s-dependent constant can be added to $\mathfrak{k}$ so that the resulting function, $\mathfrak{k}_0$, depends smoothly on s and obeys $|\frac{\partial}{\partial s} \mathfrak{k}_0| \leq c_0 \langle p_X \rangle_2 T^{-1/2}$.

Reintroduce $\sigma_\perp$ from Part 2 of Section 9.2. The 2-form $w + d(\sigma_\perp \mathfrak{q}_0)$ is equal to w on $\mathbb{R} \times \mathcal{Y}_0$ on the $r > \rho_* + \frac{3}{4}\varepsilon$ part of $\mathbb{R} \times \mathcal{N}_\varepsilon$. Meanwhile, it is equal to $w + d\mathfrak{q}$ on the $r < \rho_* + \frac{5}{8}\varepsilon$ part of $\mathbb{R} \times \mathcal{N}_\varepsilon$. Moreover, the norm of the difference between this 2-form and w on the $r > \rho_* + \frac{1}{2}\varepsilon$ part of $\mathcal{N}_\varepsilon$ is bounded by $c_0 \langle p_X \rangle_2 T^{-1/2}$, this being a consequence of the bounds in the preceding paragraph for $\mathfrak{q}_0$.

Of interest in what follows is the 2-form on $\mathbb{R} \times \mathcal{Y}_{0\varepsilon}$ given by

$$ds \wedge \mathfrak{b} + w + d(\sigma_\perp \mathfrak{q}_0) \quad \text{with } \mathfrak{b} = \upsilon_\diamond + \sigma_\perp \frac{\partial}{\partial s}\mathfrak{q}_0 + d(\sigma_\perp \frac{\partial}{\partial s}\mathfrak{k}_0). \tag{9.19}$$

This is a closed 2-form on $\mathbb{R} \times \mathcal{Y}_{0\varepsilon}$ which is $ds \wedge *w + w$ on $\mathbb{R} \times \mathcal{Y}_0$ and on the $r > \rho_* + \frac{3}{4}\varepsilon$ part of $\mathbb{R} \times \mathcal{N}_\varepsilon$. The bounds given above on the norms of $\mathfrak{k}_0$, its s-derivative, and on the norms of $\mathfrak{q}_0$, $d\mathfrak{q}_0$ and $\frac{\partial}{\partial s}\mathfrak{q}_0$ imply the following: There exists $c_\diamond > 1$ such that each $s \in \mathbb{R}$ version of the 3-form $\mathfrak{b} \wedge (w + d(\sigma \mathfrak{q}_0))$ is strictly positive on $\mathcal{Y}_{0\varepsilon}$ if

$$\langle p_X \rangle_2 T^{-1/2} \le c_\diamond^{-1}. \tag{9.20}$$

Assume in what follows that this bound holds. Granted (9.20), then Lemma 9.3 supplies a smooth, s-dependent metric on $\mathcal{Y}_{0\varepsilon}$ with the properties listed below. The notation uses $\mathfrak{g}_X$ to denote the metric at any given $s \in \mathbb{R}$.

(9.21)

- The Hodge star of $\mathfrak{g}_X$ sends $w + d(\sigma \mathfrak{q}_0)$ to $\mathfrak{b}$.
- The metric $\mathfrak{g}_X$ is $\mathfrak{g}_*$ on $\mathbb{R} \times \mathcal{Y}_0$ and on the $\mathrm{r} > \rho_* + \frac{3}{4}\varepsilon$ part of $\mathbb{R} \times \mathcal{N}_\varepsilon$.
- The metric $\mathfrak{g}_X$ is the metric in (9.7) on the $\mathrm{r} < \rho_* + \frac{5}{8}\varepsilon$ part of $\mathbb{R} \times \mathcal{N}_\varepsilon$.
- Given $k \in \{1, 2, \ldots\}$, there exists $c_k > 1$ such that the $s < -104$ and $s > 104$ versions of $\mathfrak{g}_X$ and their derivatives to order $k \ge 1$ differ by at most $c_k e^{-|s|/c_0}$ from the metric $\mathfrak{g}_{-T}$ on Y_- incarnation of $\mathcal{N}_\varepsilon$ or $\mathfrak{g}_{+T}$ on the Y_+ incarnation as the case may be.

By way of an explanation for the fourth bullet, this follows from (9.19) and the third bullet of Lemma 9.4 given the following fact: The derivatives to order k of any given coclosed eigenvector of $*d$ on Y_- or Y_+ with L^2-norm 1 is bounded by a polynomial function of the norm of the eigenvalue with coefficients that are determined solely by the given metric.

Part 3: Let $\mathfrak{m}_T$ denote the metric on X that is equal to $\mathfrak{m}$ on complement of the image of (9.10)'s embedding and whose pull-back to $\mathbb{R} \times \mathcal{Y}_{0\varepsilon}$ via this embedding is the metric $ds^2 + \mathfrak{g}_X$. This is a smooth metric on X whose pull-back by the embeddings from the

second and third bullets of (2.7) converge as $s \to -\infty$ to the metric $ds^2 + \mathfrak{g}_{-T}$ and converge as $s \to \infty$ to the metric $ds^2 + \mathfrak{g}_{+T}$. These pull backs are also independent of s for $|s| > 104$ at points of the form (s, p) if p is in either $\mathcal{Y}_0$, the $r > \rho_* + \frac{3}{4}\varepsilon$ part of $\mathcal{N}_\varepsilon$, or $\mathcal{Y}_M$.

Let ω_T denote the closed 2-form on X given by ω on the complement of the image of (9.10)'s embedding and whose pull-back to $\mathbb{R} \times \mathcal{Y}_{0\varepsilon}$ via this embedding is the 2-form in (9.19). The 2-form ω_T is closed. This 2-form is also self-dual when self-duality is defined by the metric $\mathfrak{m}_T$, this being a consequence of the first bullet in (9.21). Let w_{-T} and w_{+T} denote the $\mathfrak{g}_{-T}$ and $\mathfrak{g}_{+T}$ harmonic 2-forms with de Rham cohomology class that of $c_1(\det(\mathbb{S}))$. Use $*$ in what follows to denote either the $\mathfrak{g}_{-T}$ or $\mathfrak{g}_{+T}$ Hodge dual. The pull-backs of ω_T via the embedding from the second bullet of (2.7) differs from $ds \wedge *w_{-T} + w_{-T}$ in the C^1 topology by at most $c_T e^{-|s|/c_T}$ with $c_T > 1$ being a constant. The pull-back via the embedding from the third bullet of (2.7) differs from $ds \wedge *w_{+T} + w_{+T}$ in the C^1-topology by at most c_T . By way of an explanation, these bounds follow from the second and third bullet of Lemma 9.4. Keep in mind that ω_T obeys the second and third bullets of (2.7).

Neither ω_T nor $\mathfrak{m}_T$ are likely to be s-independent where $|s|$ is sufficiently large. This is a defect that is remedied in Parts 4-7 below.

Part 4: Both w_{-T} and w_{+T} have nondegenerate zeros on the components of Y_- and Y_+ where they are not identically zero, these being the components where $c_1(\det(\mathbb{S}))$ is not torsion. Let $Y_* \subset Y_-$ denote such a component and let $p \in Y_*$ denote a zero of w_{-T}. Let $B \subset Y_*$ denote a small radius ball centered on p with the following properties: The point p is the only zero of w_{-T} in the closure of B; and B is disjoint from $\mathcal{Y}_0$ and from the $r > \rho_* + \frac{3}{4}\varepsilon$ part of $\mathcal{N}_\varepsilon$. Since w_{-T} vanishes transversely at p, there exists $L_0 > 1$ such that each $s < -L_0$ version of $w_{-T} + dq$ vanishes transversely in the closure of B at a single point. Let p_s denote this point. Note in particular that $\operatorname{dist}(p, p_s) \le c_-$. Granted that $\operatorname{dist}(p, p_s) \ll 1$ for $s \ll -1$, there exists $s_0 > 1$ such that $\operatorname{dist}(p, p_s)$ is less than $\frac{1}{8}$ times the radius of B when $s \le -s_0$. This being the case, there exists $L_0 > s_0$, $c_- > 1$ and a family of diffeomorphisms of Y_* parametrized by $(-\infty, -L_0]$ with the properties in the list that follows. The list uses Ψ_s to denote the

diffeomorphism labeled by a given $s \in (-\infty, -L_0]$.

(9.22)
- If $s > -L_0 - 1$, then Ψ_s is the identity map.
- Every $s \in (-\infty, -L_0]$ version of Ψ_s is the identity where $\operatorname{dist}(\cdot, p) > 2 \operatorname{dist}(p, p_s)$.
- Every $s \in (-\infty, -L_0]$ version of Ψ_s differs from the identity in the C^{10}-topology by at most $e^{-|s|/c}$.
- $\Psi_s(p) = p_s$ when $s < -L_0 - 2$.

This family of diffeomorphisms defines a diffeomorphism of X which is the identity on the $s > -L_0 - 1$ part of X, and on the image in X of $(-\infty, -L_0] \times (Y_- - Y_*)$ via the diffeomorphism in the second bullet of (2.7). This diffeomorphism is defined on the image of $(-\infty, -L_0] \times Y_*$ via the second bullet of (2.7) by that of $(-\infty, -L_0] \times Y_*$ that sends a given point (s, q) to $(s, \Psi_s(q))$. Use Ψ_p to denote this diffeomorphism of X. Various versions of this diffeomorphism are defined by the zeros of w_{-T} on the components of Y_- where $c_1(\det(\mathbb{S}))$ is not torsion. These diffeomorphisms pairwise commute. Use Ψ to denote their composition.

Introduce $\mathfrak{m}_{T0}$ to denote $\Psi^*\mathfrak{m}_T$ and ω_{T0} to denote $\Psi^*\omega_T$. The 2-form ω_{T0} is closed and it is self-dual if the notion of self-duality is defined using $\mathfrak{m}_{T0}$. The form ω_{T0} can be written on $(-\infty, -L_0] \times Y_*$ as $ds \wedge (*w_{-T} + \mathfrak{n}) + (w_{-T} + \mathfrak{m})$ where $\mathfrak{n}$ and $\mathfrak{m}$ have C^1-norm less than $c_- e^{-|s|/c_-}$ and both vanish on $(-\infty, -1] \times \mathcal{Y}_0$ and on the $r > \rho_* + \frac{3}{4}\varepsilon$ part of $(-\infty, -1] \times \mathcal{N}_\varepsilon$. By way of notation, c_- denotes here and in what follows a constant that is greater than 1. Its value can increase between successive appearances. Note that the fact that ω_{T0} is closed requires that $d\mathfrak{n}$ equals $\frac{\partial}{\partial s}\mathfrak{m}$.

The pull-back of $w_{-T} + \mathfrak{m}$ to each constant s slice of $(-\infty, -1] \times Y_*$ defines the same cohomology class as w_{-T}. This implies in particular that $\mathfrak{m} = d\mathfrak{u}$ with $\mathfrak{u}$ being an s-dependent 1-form on Y_*. Any s-dependent, closed 1-form can be added to $\mathfrak{u}$ without changing $d\mathfrak{u}$, and this fact is used to choose $\mathfrak{u}$ so that the conditions that follow hold.

(9.23)
- The 1-form $\mathfrak{u}$ is zero on $\mathcal{Y}_0$ and on the $r > \rho_* + \frac{3}{4}\varepsilon$ part of $\mathcal{N}_\varepsilon$.
- The C^2 norm of $\mathfrak{u}$ is less than $c_- e^{-|s|/c_-}$.
- Let p denote a zero of w_{-T} in $(-\infty, -L_0] \times Y_*$. Then $|\mathfrak{u}|$ and the norm of $\mathfrak{u}$'s covariant derivative along $\frac{\partial}{\partial s}$ at any $s \in (-\infty, -L_0] \times Y_*$ is bounded by $c_- \operatorname{dist}(\cdot, p)^2 e^{-|s|/c_-}$.

To explain how the third bullet can be satisfied, let p again denote a zero of w_{-T}. Use the metric $\mathfrak{g}_{-T}$ to construct a Gaussian coordinate chart centered at p so as to identify

B with a small radius ball in $\mathbb{R}^3$. The corresponding coordinate map to $\mathbb{R}^3$ is denoted by x or (x_1, x_2, x_3). Write the two form ω_{T0} as $ds \wedge (*w_{-T} + \mathfrak{n}) + (w_{-T} + \mathfrak{m})$. The 2-form $\mathfrak{m}$ appears in these coordinates as

$$\mathfrak{m} = \frac{1}{2}\mathfrak{o}^{ij} x^i \varepsilon^{jnm} dx^n dx^m + \cdots \tag{9.24}$$

where the summation convention over repeated indices is used. The unwritten terms in (9.24) are $\mathcal{O}(|x|^2)$. What is denoted by $\{\varepsilon^{jnm}\}_{1\leq j,n,m\leq 3}$ is anti-symmetric with respect to interchanging indices and so defined by the rule $\varepsilon^{123} = 1$. Meanwhile, $\{\mathfrak{o}^{ij}\}_{i,j=1,2,3}$ are the entries of a traceless, s-dependent matrix whose norm and that of its s-derivative are at most $c_- e^{-|s|/c_1}$. The matrix is traceless because $\mathfrak{m}$ is closed. The fact that this matrix $\mathfrak{o}$ is traceless implies that $\mathfrak{m}$ on B can be written as du_B with $u_B = \frac{1}{6}\mathfrak{o}^{ij} x^i x^n \varepsilon^{jnm} dx^m + \cdots$ where the unwritten terms are $\mathcal{O}(|x|^3)$. Since $u - u_B = dp$ on B, it follows that u can be modified with no change near the boundary of B so that $u = u_B$ on a small radius ball in B centered at p.

Part 5: Fix $L_1 > L_0 + 1$ and let χ_1 denote the function on $\mathbb{R}$ given by $\chi(-L_1 - s)$. This function equals zero where $s < -L_1 - 1$ and it equals 1 when $s > -L_1$. Use χ_1' to denote the derivative of χ_1. The function χ_1 and the 2-form ω_{T0} are used next to define the 2-form on $(-\infty, -L_0] \times Y_*$ to be denoted by ω_{T1} . This 2-form is ω_{T0} on the $s > -L_1$ part of $(-\infty, -L_0] \times Y_*$, and it is given where $s \leq -L_1$ by the formula that follows for its pull-back via the embedding from (2.7)'s second bullet:

$$\omega_{T1} = ds \wedge (*w_{-T} + \chi_1 \mathfrak{n} + \chi_1' u) + w_{-T} + \chi_1 du. \tag{9.25}$$

The 2-form ω_{T1} is a closed 2-form on $(-\infty, -L_0] \times Y_*$. The remainder of this part of the subsection and Part 6 describe a metric on the $s \in (-\infty, -L_0] \times Y_*$ that makes ω_{T1} self-dual. This new metric is equal to $\mathfrak{m}_{T0}$ where $s \geq -L_1 + 1$ and it is equal to $ds^2 + \mathfrak{g}_{-T}$ where $s < -L_1 - 2$. This new metric is denoted below by $\mathfrak{m}_{T1}$. The five steps that follow describe the metric $\mathfrak{m}_{T1}$ at points in $(-\infty, -L_0] \times Y_*$ that project to Y_* near the zero locus of w_{-T}.

Step 1: The 2-form w_{-T} and the 1-form $*w_{-T}$ on B can be written using the Gaussian coordinates (x_1, x_2, x_3) on B as

$$w_{-T} = \frac{1}{2}\mathrm{A}^{ij} x^i \varepsilon^{jnm} dx^n dx^m + \cdots \quad \text{and} \quad *w_{-T} = \mathrm{A}^{ij} x^i dx^j + \cdots \tag{9.26}$$

with summations over repeated indices implicit. The various $i, j \in \{1, 2, 3\}$ versions of A^{ij} in (9.26) are the entries of an invertible matrix, this denoted by A. The unwritten

terms in (9.26) vanish to order $|x|^2$. The fact that w_{-T} is closed implies that A is traceless and the fact that $*w_{-T}$ is self-dual implies that A is symmetric.

The unwritten terms in (9.26) are incorporated using the notation whereby w_{-T} and $*w_{-T}$ on $(-\infty, -L_0] \times B$ are written as

$$ds \wedge (f_1\hat{e}^1 + f_2\hat{e}^2 + f_3\hat{e}^3) + f_1\hat{e}^2 \wedge \hat{e}^3 + f_2\hat{e}^3 \wedge \hat{e}^1 + f_3\hat{e}^1 \wedge \hat{e}^2, \tag{9.27}$$

where $\{\hat{e}^k\}_{1\le k\le 3}$ denotes a $\mathfrak{g}_{-T}$-orthonormal set for T^*B with $\{\hat{e}^k = dx^k + \mathcal{O}(|x|^2)\}_{1\le k\le 3}$; and where $\{f_k\}_{1\le k\le 3}$ are functions with $\{f_k = \sum_{1\le i\le 3} \mathrm{A}^{ik}x^i + \mathcal{O}(|x|^2)\}_{1\le k\le 3}$. Note in particular that these are such that $df_1 \wedge df_2 \wedge df_3 > \frac{1}{2}\det(\mathrm{A})$ on a concentric ball in B centered at the origin. This ball is denoted by B'. It is assumed in what follows that L_0 is chosen so that $\omega_{T0} = \omega_T$ on the complement of a concentric ball in B' with radius one fourth that of B'. In particular, it is assumed that (9.22)'s diffeomorphism Ψ_s is the identity for all s on a neighborhood in B of $B - B'$.

Step 2: The Ψ-pull back of $\{ds, \hat{e}^1, \hat{e}^2, \hat{e}^3\}$ is $\mathfrak{m}_T$-orthonomormal. The Ψ-pull back of ds is ds. Meanwhile, Ψ can be chosen so that

$$\Psi^*\hat{e}^k = \hat{e}^k + \sum_{1\le k\le 3} \mathfrak{p}^k ds + \sum_{1\le j\le 3} \mathfrak{p}^{kj}\hat{e}^j, \tag{9.28}$$

where $\sum_{1\le k\le 3} |\mathfrak{p}^k| \le c_- e^{-|s|/c_-}$ and $\sum_{1\le k,j\le 3} |\mathfrak{p}^{kj}| \le c_-|x|e^{-|s|/c_-}$ when $s < -L_0 - 1$. This is done by defining (9.22)'s diffeomorphism Ψ_s using the Gaussian coordinates in (9.25) by the rule $x \mapsto \Psi_s(x) = x + p_s$ at points (s, x) with $|x| < \frac{3}{2}|p_s|$ and $s < -L_0 - 1$. Use $\{\hat{e}^k_s\}_{1\le k\le 3}$ to denote $\{\Psi^*\hat{e}^k\}_{1\le k\le 3}$. Granted this notation, the 2-form ω'_{T0} near p can be written as

$$\omega'_{T0} = ds \wedge (f_{s_1}\hat{e}^1_s + f_{s_2}\hat{e}^2_s + f_{s_3}\hat{e}^3_s) + f_{s_1}\hat{e}^2_s \wedge \hat{e}^3_s + f_{s_2}\hat{e}^3_s \wedge \hat{e}^1_s + f_{s_3}\hat{e}^1_s \wedge \hat{e}^2_s, \tag{9.29}$$

where $\{f_{s_k} = \Psi^*_s f_k\}_{1\le k\le 3}$.

Step 3: Introduce $\{e^k_{s_\chi} = \hat{e}^k + \chi_0 \sum_{1\le j\le 3} \mathfrak{p}^{kj}\hat{e}^j\}_{1\le k\le 3}$. Use this s-dependent basis to write the (9.25)'s 2-form $w_{-T} + \chi_1 du$ on B' as

$$w_{-T} + \chi_1 du = f_{s_\chi 1}e^2_{s_\chi} \wedge e^3_{s_\chi} + f_{s_\chi 2}e^3_{s_\chi} \wedge e^1_{s_\chi} + f_{s_\chi 3}e^1_{s_\chi} \wedge e^2_{s_\chi}, \tag{9.30}$$

where $\{f_{s_\chi k}\}_{1\le k\le 3}$ are smoothly varying functions of s and the coordinate x with the property that $f_{s_\chi(\cdot)} = f_{(\cdot)}$ when $s < -L_1 - 1$ and $f_{s_\chi(\cdot)} = f_{s(\cdot)}$ when $s > -L_1$. This depiction can be derived from the fact that $\{f_k\}_{1\le k\le 3}$ generate $C^\infty(B')$. Note that $f_{s_\chi k} = f_k + \cdots$ with the unwritten terms such that their norms are bounded by $c_- e^{-|s|/c_-}|x|$ and such that their first derivatives have norms bounded by $c_- e^{-|s|/c_-}$.

This implies in particular that the functions $\{f_{s_\chi k}\}_{1\le k\le 3}$ also generate $C^\infty(B')$ and that $df_{s_\chi 1}\wedge df_{s_\chi 2}\wedge df_{s_\chi 3} > \det(\mathrm{A})$ on B' when $L_0 > c_-$.

The 1-form $*w_{-T}+\chi_0\mathfrak{n}+\chi_0'\mathfrak{u}$ can be written schematically on $(-\infty,-L_0]\times B'$ using the basis $\{e^k_{s_\chi}\}_{1\le k\le 3}$ as

$$*\,w_{-T}+\chi_0\mathfrak{n}+\chi_0'\mathfrak{u} = \sum_{1\le k,i\le 3} f_{s_\chi k}\mathrm{C}_{ki}\, e^i_{s_\chi}, \tag{9.31}$$

with $\{\mathrm{C}_{ki}\}_{1\le i,k\le 3}$ denoting a matrix of smooth functions of s and the coordinate x. Given that the functions $\{f_{s_\chi k}\}_{1\le k\le 3}$ also generate $C^\infty(B')$, such a depiction follows because both $\mathfrak{n}$ and $\mathfrak{u}$ vanish at p. Keep in mind for what follows that the matrix with coefficients $\{\mathrm{C}_{ki}\}_{1\le i,k\le 3}$ differs from the identity matrix by at most $c_-e^{-|s|/c_-}$.

Step 4: A particular set of three smooth functions of $s\in(-\infty,-L_0]$ and the coordinate x is specified momentarily. Let $\{\mathfrak{q}^k\}_{1\le k\le 3}$ denote any given set of such functions. Use this set to define 1-forms $\{\hat{e}^k_{s_\chi}\}_{1\le k\le 3}$ on $(-\infty,-L_0]\times B'$ by the rule

$$\hat{e}^k_{s_\chi} = e^k_{s_\chi} - \mathfrak{q}^k ds. \tag{9.32}$$

Given the formula in (9.31) and (9.32), it follows that ω_{T1} on $(-\infty,-L_0]\times B'$ can be written using $\{\hat{e}^k_{s_\chi}\}_{1\le k\le 3}$ as

$$ds\wedge\big(f_{s_\chi k}(\mathrm{C}_{ki}+\varepsilon^{kni}\mathfrak{q}^n)\big)\hat{e}^i_{s_\chi} + \frac{1}{2} f_{s_\chi k}\varepsilon^{knm}\hat{e}^n_{s_\chi}\wedge\hat{e}^m_{s_\chi}. \tag{9.33}$$

This equation uses the summation convention over repeated indices.

Step 5: The set $\{\mathfrak{q}^k\}_{1\le k\le 3}$ is introduced for the following reason: There is a unique choice for $\{\mathfrak{q}^k\}_{1\le k\le 3}$ that makes the matrix with entries $\{\mathrm{C}_{ki}+\varepsilon^{kni}\mathfrak{q}^n\}_{1\le i,k\le 3}$ a symmetric matrix, this being $\{\mathfrak{q}^k=\frac{1}{2}\varepsilon^{kin}\mathrm{C}_{ni}\}$. This choice is used in what follows. With this choice understood, a metric is defined on $(-\infty,-L_0]\times B'$ by the following rules:

$$\begin{cases} \bullet\ \ ds \text{ has norm 1 and it is orthogonal to } \{\hat{e}^{ks_\chi}\}_{1\le k\le 3}. \\ \bullet\ \ \text{Given } (i,k)\in\{1,2,3\}, \text{ then the inner product between } \hat{e}^k_{s_\chi} \text{ and } \hat{e}^i_{s_\chi} \text{ is } \mathrm{C}_{ki}+\varepsilon^{kni}\mathfrak{q}^n. \end{cases} \tag{9.34}$$

The inner product defined by the second bullet is positive definite if $L_0 > c_0c_-$ because of the afore-mentioned fact that the matrix defined by $\{\mathrm{C}_{ki}\}_{1\le i,k\le 3}$ differs by at most $c_-e^{-|s|/c_-}$ from the identity matrix.

The metric just defined is the metric $\mathfrak{m}_{T0}$ when $s > -L_1$; and it is the metric $ds^2 + \mathfrak{g}_{-T}$ when $s < -L_1 - 1$. Moreover, the 2-form ω_{T1} is self-dual on $(-\infty, -L_0] \times B'$ when self-duality is defined by this metric. Denote this metric by $\mathfrak{m}_{T1p}$.

Let $B'' \subset B'$ denote the concentric ball whose radius is one half that of B'. The desired metric $\mathfrak{m}_{T1}$ is defined to equal $\mathfrak{m}_{T1p}$ on $(-\infty, -L_0] \times B''$.

Part 6: Use U to denote the union of the various versions of the ball B''. The two steps that follow directly describe the metric $\mathfrak{m}_{T1}$ on $(-\infty, -L_0] \times (Y_* - U)$.

Step 1: This step describes a metric on $(-\infty, -L_0] \times (Y_* - U)$ to be denoted by $\mathfrak{m}_{T1\diamond}$. The metrics $\mathfrak{m}_{T1}$ and $\mathfrak{m}_{T1\diamond}$ agree on the product of $(-\infty, -L_0]$ with the complement in Y_* of the union of the various versions of the ball B'. The definition of this metric $\mathfrak{m}_{T1\diamond}$ assumes that $L_0 > c_\diamond$ with $c_\diamond$ such that $\omega = w_{-T} + \chi_1 du$ and $\upsilon = *w_{-T} + \chi_1 n + \chi_1' u$ from (9.25) obey $\upsilon \wedge \omega > 1/c_\diamond$ on $(-\infty, -L_0] \times (Y_* - U)$. The existence of $c_\diamond$ follows from (9.23). Let p denote a zero of w_{-T} and let $B_\diamond \subset B'$ denote the concentric ball whose radius is three quarters that of B'. Use V to denote the union of the various versions of $B_\diamond$. Invoke Lemma 9.3 on $(-\infty, -L_0] \times (Y_* - U)$ using ω and υ to obtain a smooth family of metrics on $Y_* - V$ parametrized by $(-\infty, -L_0]$ with the properties listed in the upcoming (9.34). The notation uses $\mathfrak{g}_\diamond$ to denote any given $s \in (-\infty, -L_0]$ member of the family. To explain more of the notation, note first that pull-backs of $\mathfrak{m}$ and Part 4's metric $\mathfrak{m}_{T0}$ via the embedding from the second bullet of (2.7) agree on $(-\infty, -L_0] \times (Y_* - U)$. In particular, the pull-back of $\mathfrak{m}_{T0}$ to this part of $(-\infty, -L_0] \times Y_*$ can be written as $ds^2 + \mathfrak{g}_X$ with $\mathfrak{g}_X$ denoting here a smooth, s-dependent metric on $Y_* - U$. This metric $\mathfrak{g}_X$ is the metric $\mathfrak{g}_{-T}$ on $\mathcal{Y}_M - U$ and it is the metric from (9.19) on $\mathcal{Y}_{0\varepsilon}$.

(9.35)
- Each $s \in (-\infty, -L_1 - 1]$ version of $\mathfrak{g}_\diamond$ is $\mathfrak{g}_{-T}$ and each $s \in [-L_1, -L_0]$ version is the corresponding version of $\mathfrak{g}_X$.
- The $\mathfrak{g}_X$-Hodge dual of the 2-form $w_{-T} + \chi_1 du$ on $Y_* - V$ is the 1-form $*w_{-T} + \chi_1 n + \chi_1' u$.

The metric $\mathfrak{m}_{T1\diamond}$ on $(-\infty, -L_0] \times (Y_* - U)$ is defined to be $ds^2 + \mathfrak{g}_\diamond$. It follows directly from the second bullet in (9.35) that the 2-form ω_{T1} is self-dual on $(-\infty, -L_0] \times (Y_* - V)$ when the notion of self duality is defined using the metric $\mathfrak{m}_{T1\diamond}$.

Step 2: Let p denote a zero of w_{-T}. The metrics $\mathfrak{m}_{T1\diamond}$ and $\mathfrak{m}_{T1p}$ are both metrics on $(-\infty, -L_0] \times (B' - B_\diamond)$. The 2-form ω_{T1} is self-dual on $(-\infty, -L_0] \times (B' - B_\diamond)$ when the latter notion is defined by either metric. Use $z_\diamond$ and z_p to denote the respective $\mathfrak{m}_{T1\diamond}$ and $\mathfrak{m}_{T1p}$ norms of ω_{T1}. Since $\omega_{T1} \wedge \omega_{T1} > c_-^{-1}$ here, there is a ω_{T1}-compatible almost complex structures for $(-\infty, -L_0] \times (B' - B_\diamond)$, these denoted by $J_\diamond$ and J_p, such that

$$\mathfrak{m}_{T1\diamond} = z_\diamond^{-1}\omega_{T1}(\cdot, J_\diamond(\cdot)) \quad \text{and} \quad \mathfrak{m}_{T1p} = z_p^{-1}\omega_{T1}(\cdot, J_p(\cdot)). \tag{9.36}$$

As the space of ω_{T1}-compatible almost complex structures on $(-\infty, -L_0] \times (B' - B_\diamond)$ is contractible, there exists such an almost complex structure with two properties, the first of which is as follows: The almost complex structure is J_p at points with $B' - B_\diamond$ component in a neighborhood of the boundary of the closure of $B_\diamond$; and it is $J_\diamond$ at points with $B' - B_\diamond$ component in the B' part of a neighborhood of the boundary of the closure of B' in B. To state the second property, keep in mind that $J_\diamond = J_p$ in some neighborhood of $(-\infty, -L_1 - 1] \times (B' - B_\diamond)$ and also in some neighborhood of $[-L_1, -L_0] \times (B' - B_\diamond)$. What follows is the second property: The new almost complex structure is $J_\diamond$ and thus J_p in slightly smaller neighborhood of $(-\infty, -L_1 - 1] \times (B' - B_\diamond)$ and $[-L_1, -L_0] \times (B' - B_\diamond)$. Use J_* to denote an almost complex structure of the sort just described.

Fix a smooth, strictly positive function on $(-\infty, -L_1 - 1] \times (B' - B_\diamond)$ that is equal to $z_\diamond$ where $J_* = J_\diamond$ and equal to z_p where $J_* = J_p$. Denote this function by z_*. Use J_* and z_* to define the metric $\mathfrak{m}_{T1}$ on $(-\infty, -L_1 - 1] \times (B' - B_\diamond)$ by the rule $\mathfrak{m}_{T1} = z_*^{-1}\omega_{T1}(\cdot, J_*(\cdot))$. This metric smoothly extends the metrics defined in Step 1 and in Step 5 of Part 5 and it has all of the desired properties.

Part 7: Let $Y_* \subset Y_-$ now denote a component where w_{-T} is identically zero, thus a component where $c_1(\det(\mathbb{S}))$ is torsion. Suppose that $L > 1$ has been chosen. Let ω_{T0} now denote the pull-back of ω_T to $(-\infty, -L] \times Y_*$ via the embedding from the second bullet of (2.7). It follows from Lemma 9.4 that the C^1 norm of ω_{T0} is bounded by $c_0\langle p_X\rangle_2 e^{-|s|/c_0}$. The 2-form ω_{T0} is exact on $(-\infty, -L] \times Y_*$, it can be written as $ds \wedge \frac{\partial}{\partial s}u + du$ with d denoting here the exterior derivative along the constant s slices of $(-\infty, -L] \times Y_*$ and with u denoting a smooth, s-dependent 1-form on Y_* with $|u|$, $|du|$ and $|u|$ bounded by $c_0\langle p_X\rangle_2\, e^{-|s|/c_0}$.

With the preceding understood, fix $L_{tor} > L + 4$ and let χ_* denote the function on $\mathbb{R}$ defined by the rule $s \mapsto \chi(-L_{tor}+3-s)$. This function equals 1 where $s > -L_{tor}+3$ and it equals 0 where $s < -L_{tor} + 2$. Use χ_* to define a self-dual form on $(-\infty, -L] \times Y_*$ by the following rules: This form is equal to ω_{T0} on $[-\infty, -L_{tor} + 4, -L] \times Y_*$,

it is identically 0 on $[-\infty, -L_{tor}] \times Y_*$ and it is equal to $\chi_*(ds \wedge \frac{\partial}{\partial s}u + du)$ on $[-L_{tor}, -L_{tor}+4] \times Y_*$. Denote this 2-form by ω_{T1}.

The 2-form ω_{T1} can be written as $ds \wedge *w_* + w_*$ with $w_* = d(\chi_* u)$ with it understood again that d here denotes the exterior derivative along Y_*. Let χ_*' denote the derivative of the function $s \mapsto \chi_*(s)$. The 2-form w_* on $[-L_{tor}, -L_{tor}+4] \times Y_*$ can be written as $d\mathfrak{b}$ with $\mathfrak{b} = \chi_*' u + \chi_* u$. Note in particular that $|\mathfrak{b}| \le c_0 c \langle p_X \rangle_2 e^{-|s|/c_0}$.

Fix $c > 1$. The bound just given for $|\mathfrak{b}|$ leads to the following conclusion: Fix $\mathrm{r} > 1$. Then $|\mathfrak{b}| \le \mathrm{r}^{-10}$ if $L_{tor} > c_0(|\ln(\langle p_X \rangle_2| + \ln \mathrm{r})$.

Part 8: Define the 2-form ω_{T*} on the $s \le 0$ part of X as follows: This 2-form is equal to ω_T where $s \in [-L, 0]$. Meanwhile, its pull-back to each component of the $s < -1$ part of X via the embedding from the second bullet of (2.7) is the corresponding version of the 2-form ω_{T1}. Modulo notation, what is said in Parts 4-7 can be repeated for the $s > 0$ part of X to extend the definition of ω_{T*} and the metric $\mathfrak{m}_{T*}$ to the whole of X. The form ω_{T*} is self-dual if the latter notion is defined by $\mathfrak{m}_{T*}$. This construction has the following additional property: Suppose that p_X obeys (9.20). Fix $c > c_0$. If $r > 1$ has been chosen to be greater than a purely c-dependent constant, then the $(L = c, L_{tor} = c \ln \mathrm{r})$ version of $\mathfrak{m}_{T*}$ and ω_{T*} obey the constraints given by (2.8), (2.11), (3.11), (3.12) and the (c, r) version of (3.13). Here, the closed 1-form υ_X can be chosen so that it is s-independent and $\upsilon_X = *w_{\pm T}$ over constant s-slices of X where $|s| > L - 4$. The bounds in items 4b), 4d), and 5c) of (3.13) follows from the bounds on u in (9.23) and those for $\mathfrak{b}$ in Part 7 above.

Part 9: The happy conclusions of Part 8 are contingent on the existence of a closed 2-form, p_X, on X with the following properties: The de Rham cohomology class of p_X is $c_1(\det(\mathbb{S}))$, it equals w_- where $s < -102$, it equals w_+ where $s > 102$, and it obeys the bound in (9.20).

The subsequent four steps in this part of the subsection construct p_X on various parts of X. These constructions are used in Part 11 and they are also used in the proofs of Proposition 3.12 and 3.13.

Step 1: This step first states and then proves a lemma that supplies a crucial tool for what is to come.

Lemma 9.5 *Let U denote a 3-manifold and let $V \subset U$ denote an open set with compact closure in V. Given the data set consisting of U, V, and a Riemannian metric*

on U, there exists $\kappa > 1$ with the following significance: Let u denote a closed, exact 2-form on U. There is a 1-form on U, this denoted by q, with $\int_V |q|^2 \leq \kappa \int_U |u|^2$ and such that $dq = u$.

To set the notation used below, the L^2-norm of a function or differential form over a given set $W \subset U$ is denoted by $\|\cdot\|_W$.

Proof. The set V has a finite cover by Gaussian coordinate balls with centers in U with the property that the mutual intersection of balls from this cover is either empty or convex. This cover can also be chosen so that each ball has the same radius and such that no ball intersects more than c_0 others. The minimal number of balls in such a cover, their common radius and the combinatorical properties of the mutual intersections is determined a priori by U, V and the metric. Let $\mathfrak{U}$ denote such a cover and let σ denote the radius of its constituent balls.

Let $B \in \mathfrak{U}$. The fact that B is convex can be used to write u on B as $u = dq_B$ where $\|q_B\|_B \leq c_0\sigma\|u\|_B$. Let B and B' denote two sets from $\mathfrak{U}$. Then $dq_B - dq_{B'} = 0$ on their intersection, and so $q_B - q_{B'} = d\mathit{k}_{BB'}$ with $\mathit{k}_{BB'}$ being a function on $B' \cap B$. It follows that $\|d\mathit{k}_{BB'}\|_{B'\cap B} \leq c_0\sigma(\|u\|_B + \|u\|_{B'})$. Changing $\mathit{k}_{BB'}$ by a constant if needed produces a version with $\|\mathit{k}_{BB'}\|_{B\cap B'} \leq c_0\sigma\|d\mathit{k}_{BB'}\|_{B'\cap B}$ and thus $\|d\mathit{k}_{BB'}\|_{B'\cap B} \leq c_0\sigma^2(\|u\|_B + \|u\|_{B'})$.

Now suppose that B, B', and B'' are from $\mathfrak{U}$ with a point in common. Let $c_{BB'B''}$ denote $\mathit{k}_{BB'} + \mathit{k}_{B'B''} + \mathit{k}_{B''B}$. This $c_{BB'B''}$ is constant and the collection of such numbers is a Čech cohomology cocycle whose cohomology class gives the class of u via the de Rham isomorphism. It follows that this cocycle is zero, and so $c_{BB'B''} = c_{BB'} + c_{B'B''} + c_{B''B}$ with each term being constant. Noting that $|c_{BB'B''}| \leq c_0\sigma^{-1}(\|u\|_B + \|u\|_{B'} + \|u\|_{B''})$, it follows that $|c_{BB'}| \leq c_*\sigma^{-1}\sup_{B''\in U: B''\cap B'\cap B\neq\emptyset}(\|u\|_B + \|u\|_{B'} + \|u\|_{B''})$ with $c_* \geq 1$ determined a priori by the combinatorics of the cover $\mathfrak{U}$.

Let $\{\chi_B\}_{B\in\mathfrak{U}}$ denote a partition of unity subbordinate to the cover $\mathfrak{U}$. Note that these functions can be chosen so that $|d\chi_B| \leq c_0\sigma^{-1}$. Define now a 1-form q on B by the rule $q|_B = q_B + d(\sum_{B'} \chi_{B'}(\mathit{k}_{BB'} - c_{BB'}))$. This defines a smooth 1-form on V with $dq = u$ and with $\|q\|_V \leq c_*\sigma\|u\|_U$.

Step 2: This lemma that is stated and then proved in this step makes the first application of Lemma 9.5.

Lemma 9.6 *There exists $\kappa > 0$ with the following significance: Fix $k \in \{0, \ldots, \mathrm{G}\}$ and then $T > 1$ so as to define Met_T on Y_k. Let $\mathfrak{g}$ denote a Met_T metric on Y_k and*

let $w_{\mathfrak{g}}$ denote the corresponding harmonic 2-form whose de Rham cohomology class is that of $c_1(\det(\mathbb{S}))$. The 2-form $w_{\mathfrak{g}}$ on the $r \in [\rho_* - \frac{1}{16}\varepsilon, \rho_* + \frac{1}{128}\varepsilon]$ part of $\mathcal{N}_\varepsilon$ can be written as dq with q being a 1-form whose L^2-norm on this part of $\mathcal{N}_\varepsilon$ is bounded by κ/T times that of $w_{\mathfrak{g}}$.

Proof. The metric on the $r \in [\rho_* - \frac{1}{8}\varepsilon, \rho_* + \frac{1}{64}\varepsilon]$ part of $\mathcal{N}_\varepsilon$ is the metric given by (9.7) with $\rho_T = \rho/T$ and with $x_{3T} = x_3/T$. The functions K and h are smooth around $\rho = 0$ with $h(0)$ and $\mathrm{K}(0) = 1$. It follows as a consequence that the metric in the region of interest when written using ρ_T and x_T is uniformly close for $T > c_0$ to the Euclidean metric on the part of the radius $(\rho_* + \frac{1}{64}\varepsilon)/T$ ball about the origin in $\mathbb{R}^3$ that lies outside the concentric ball of radius $(\rho_* - \frac{1}{8}\varepsilon)/T$. Take this to be the region U for Lemma 9.5 and take V to be the part of this same ball where the radius is between $(\rho_* - \frac{1}{16}\varepsilon)/T$ and $(\rho_* + \frac{1}{128}\varepsilon)/T$. A cover $\mathfrak{U}$ can be found as in the proof of Lemma 9.5 with a T-independent bound on the number of sets, a T-independent combinatorical structure to the intersections between them, and a common radius for the balls, c_0. This can be done because the T-dependence is just given by scaling the coordinates. Granted all of this, then the claim by the lemma follows by appeal to Lemma 9.5. $\square$

Step 3: This step supplies a part of what will be p_X on the $s \in [-102, -101]$ part of X when Y_- is a $k \in \{0, \ldots, \mathrm{G}\}$ version of Y_k, and on the $s \in [100, 102]$ part of X when Y_+ is a $k \in \{0, \ldots, \mathrm{G}\}$ version of Y_k. The constructions that follow use the embeddings from the second and third bullets of (2.7) to view the $s < 0$ and $s > 0$ parts of X as $(-\infty, 0) \times Y_-$ and as $(0, \infty) \times Y_+$.

Let $\chi_{\diamond 1}$ denote the function on $\mathbb{R}$ given by the rule $\chi(|s| - 101)$. Denote its derivative by $\chi'_{\diamond 1}$. This function is equal to 0 where $|s| \geq 102$ and it is equal to 1 where $|s| \leq 101$. Use χ to construct a smooth function on $\mathcal{N}_\varepsilon$ that equals 0 where $|r - \rho_*| > \frac{1}{128}\varepsilon$ and equals 1 where $|r - \rho_*| < \frac{1}{256}\varepsilon$. Construct this function of r so that its derivative is bounded by c_0. Use σ_1 to denote this new function of r.

If Y_- is a $k \in \{0, \ldots, \mathrm{G}\}$ version of Y_k, let q_{1-} denote the $w_{\mathfrak{g}} = w_-$ version of q that is given by Lemma 9.6. Define $p_{\mathcal{N}1}$ where $s \in [-102, -101]$ to be

$$p_{\mathcal{N}1} = -ds \wedge \chi'_{\diamond 1}\sigma_1 q_{1-} + w_- - \chi_{\diamond 1} d(\sigma_1 q_{1-}). \tag{9.37}$$

This is a closed form with de Rham cohomology class that of $c_1(\det(\mathbb{S}))$ and it equals w_- where $s \leq -102$. Of particular note is the fact that $p_{\mathcal{N}1} = 0$ on the $|r - \rho_*| < \frac{1}{256}\varepsilon$ part of $\mathcal{N}_\varepsilon$ where $s > -101$ and that it equals w_- on the complement of the $|r - \rho_*| < \frac{1}{128}\varepsilon$

part of $\mathcal{N}_\varepsilon$. It follows from Lemma 9.6 that the L^2 norm of $p_{\mathcal{N}1}$ at any given $s \in [-102, -101]$ is bounded by c_0 times that of w_-.

If Y_+ is a $k \in \{0, \ldots, \mathrm{G}\}$ version of Y_k, then very much the same formula defines an $s \in [101, 102]$ analog to $p_{\mathcal{N}1}$. The latter is obtained by using Lemma 9.6 with $w_{\mathfrak{g}} = w_+$. Lemma 9.6 supplies a 1-form q_{1+} with $dq_{1+} = w_+$ on the $|r - \rho_*| < \frac{1}{128}\varepsilon$ part of $\mathcal{N}_\varepsilon$. Use w_+ and q_{1+} in (9.37) in lieu of w_- and q_- to define $p_{\mathcal{N}1}$ where $s \in [101, 102]$.

Step 4: This step extends the definition of $p_{\mathcal{N}1}$ to the $s \in [-101, -100]$ part of X when Y_- is a $k \in \{0, \ldots, \mathrm{G}\}$ version of Y_k, and to the $s \in [100, 101]$ part of X when Y_+ is a $k \in \{0, \ldots, \mathrm{G}\}$ version of Y_k. The embeddings from the second and third bullets of (2.7) are again used to view the $s < 0$ and $s > 0$ parts of X as $(-\infty, 0) \times Y_-$ and as $(0, \infty) \times Y_+$.

Thie extension of $p_{\mathcal{N}1}$ uses the function $\chi_{\diamond 2}$ on $\mathbb{R}$ that is given by $\chi(|s| - 100)$. The latter function is 0 where $|s| \geq 101$ and it is equal to 1 where $|s| \leq 100$. The derivative of $\chi_{\diamond 2}$ is denoted by $\chi'_{\diamond 2}$. Reintroduce the closed 2-form p_0 from Step 1 of the proof of Lemma 9.1. By way of a reminder, this 2-form has compact support on $\mathcal{Y}_0$; and it has integral 2 over each cross sectional 2-sphere in $\mathcal{H}_0$.

Suppose that Y_- is a $k \in \{0, \ldots, \mathrm{G}\}$ version of Y_k. The extension of $p_{\mathcal{N}1}$ will equal $p_{\mathcal{N}1}$ on the complement in Y_- of the union of $\mathcal{Y}_0$ and the $r \geq \rho_* + \frac{1}{512}\varepsilon$ part of $\mathcal{N}_\varepsilon$. Lemma 9.5 is used momentarily to obtain a 1-form to be denoted by q_{2-} with the following properties: The 1-form q_{2-} has compact support on $\mathcal{Y}_0$ and the $r \geq \rho_* + \frac{1}{512}\varepsilon$ part of $\mathcal{N}_\varepsilon$, its exterior derivative is $w_{\mathfrak{g}} = w_- - p_0 + d(\sigma_1 q_{1-})$, and its L^2-norm is bounded by c_0 times that of w_-. Granted such a 1-form, the extension of $p_{\mathcal{N}1}$ is given by

$$p_{\mathcal{N}2} = -ds \wedge \chi'_{\diamond 2} q_{2-} + w_- - d(\sigma_1 q_{1-}) + \chi_{\diamond 2}\, dq_{2-}. \tag{9.38}$$

This is a closed 2-form that equals $p_{\mathcal{N}1}$ where $s \leq -101$ and for all $s \in [-101, -100]$ on the complement of $\mathcal{Y}_0$ and the $r \geq \rho_* + \frac{1}{512}\varepsilon$ part of $\mathcal{N}_\varepsilon$. This 2-form for $s \geq -100$ is equal to p_0 on $\mathcal{Y}_0$ and the $r \geq \rho_*$ part of $\mathcal{N}_\varepsilon$.

The application of Lemma 9.5 takes $U = V = S^1 \times S^2$. The diffeomorphism Φ_T in Part 6 of Section 9.1 is used to view $p_0 - (w_- - d(\sigma_1 q_{1-}))$ as a smooth 2-form on $S^1 \times S^2$, and viewed as such, Lemma 9.5 is applied using this 2-form for $w_{\mathfrak{g}}$. Lemma 9.5 then finds a 1-form, q, on $S^1 \times S^2$ with $dq = p_0 - (w_- - d(\sigma_1 q_{1-}))$ and with L^2-norm bounded by c_0 times the L^2-norm of w_- on Y_-. The next two paragraphs explains how to obtain q_{2-} from q.

View $\mathfrak{p}_0 - (w_- - d(\sigma_1 \mathfrak{q}_{1-}))$ as a 2-form on $S^1 \times S^2$ as done in the preceding paragraph. As explaind in Part 4 of Section 9.1, the coordinates (ρ, ϕ, x_3) for $\mathcal{N}_\varepsilon$ can be viewed where $r \leq \rho_* + \frac{1}{16}\varepsilon$ as coordinates for a ball of this same radius in $S^1 \times S^2$. The 2-form $\mathfrak{p}_0 - (w_- - d(\sigma_1 \mathfrak{q}_{1-}))$ vanishes on the concentric ball of radius $(\rho_* + \frac{1}{256}\varepsilon)/T$. It follows as a consequence that $\mathfrak{q}$ can be written as $d\mathfrak{k}$ with $\mathfrak{k}$ being a smooth function on this ball. Since the L^2 norm of $d\mathfrak{k}$ on this ball is bounded by c_0 times the L^2-norm of w_- over Y_-, it follows that $\mathfrak{k}$ can be modified by adding a constant if nessecary so that its L^2-norm over this ball is bounded by c_0/T times the L^2-norm of w_- over Y_-.

Use χ to construct a smooth function of the radial coordinate on this ball with compact support that equals 1 on the concentric ball of radius $(\rho_* + \frac{1}{512}\varepsilon)/T$ ball. In particular, such a function can be constructed so that the absolute value of its derivative is bounded by $c_0 T$. Let σ_2 denote such a function and define $\mathfrak{q}_*$ to be $\mathfrak{q} - d(\sigma\mathfrak{k})$. This 1-form has the same properties as $\mathfrak{q}$ but it is zero on the complement of the image of the embedding Φ_T from Part 6 of Section 9.1. The desired 1-form $\mathfrak{q}_{2-}$ is $\Phi_T^* \mathfrak{q}_*$.

If Y_+ is either a $k \in \{0, \ldots, \mathrm{G}\}$ version of Y_k, then there is an analogous construction that defines $\mathfrak{p}_{\mathcal{N}2}$ on the $s \in [100, 101]$ part of X. The formula for the latter is given by replacing w_-, $\mathfrak{q}_{1-}$ and $\mathfrak{q}_{2+}$ by w_+, $\mathfrak{q}_{1+}$ and a 1-form, $\mathfrak{q}_{2+}$, that is defined by the rules given in the preceding paragraph with w_+ and $\mathfrak{q}_{1+}$ used in lieu of w_- and $\mathfrak{q}_{1-}$.

Part 10: Constructions in Part 11 and in the proof of Proposition 3.12 require a particular choice for the metric $\mathfrak{m}$ on certain parts of X. The constraint given momentarily holds on the $s \in [-100, -96]$ part of X when Y_- is a $k \in \{0, \ldots, \mathrm{G}\}$ version of Y_k, and it holds on the $s \in [96, 100]$ part of X when Y_+ is a $k \in \{0, \ldots, \mathrm{G}\}$ version of Y_k.

The statement of the constraint uses the embeddings from the second and third bullets of (2.7) to view the $s < 0$ and $s > 0$ part of X as $(-\infty, 0] \times Y_-$ and as $(0, \infty) \times Y_+$. Viewed this way, the constraint on the metric $\mathfrak{m}$ involves only the $r \in [\rho_* - \frac{15}{16}\varepsilon, \rho_*)$ parts of $[-100, -96] \times \mathcal{N}_\varepsilon$ and $[96, 100] \times \mathcal{N}_\varepsilon$. To define $\mathfrak{m}$ on these parts of X, construct a smooth, non-decreasing function on $\mathbb{R}$ to be denoted by $T_\diamond$: This function equals T where $|s| \geq 99$ and it equals 1 where $|s| \leq 98$. The ubiquitous function χ can be used to define this function $T_\diamond$. Reintroduce the metric $\mathfrak{g}_T$ on $\mathcal{N}_\varepsilon$ that is defined in Part 5 of Section 9.1. The assignment $s \mapsto \mathfrak{g}_{T_\diamond}$ defines a 1-parameter family of metrics on $\mathcal{N}_\varepsilon$ with parameter space either $[-100, -96]$ or $[96, 100]$ as the case may be. The $|s| = 100$ end member of this family is $\mathfrak{g}_T$ and the $|s| = 96$ member is the metric in (9.6).

Use χ to construct a smooth function of the coordinate r on $\mathcal{N}_\varepsilon$ that is equal to 1 where $r < \rho_* - \frac{1}{1024}\varepsilon$ and equal to 0 where $r > \rho_* - \frac{1}{2048}\varepsilon$. Use σ_* to denote this function.

The metric $\mathfrak{m}$ is constrained by requiring that its pull-back to $[-100,-96]\times\mathcal{N}_\varepsilon$ via the embedding from the second bullet of (2.7) or to $[96,100]\times\mathcal{N}_\varepsilon$ via the embedding from the third bullet of (2.7) to be the metric

$$ds^2 + \sigma_*\,\mathfrak{g}_{T_\diamond} + (1-\sigma_*)\,\mathfrak{g}_T. \tag{9.39}$$

Note in particular that this metric smoothly extends $ds^2+\mathfrak{g}_T$ near $|s|=100$ and it smoothly extends $ds^2+\mathfrak{g}_T$ from the $r\le\rho_*-\frac{15}{16}\varepsilon$ part of $\mathcal{N}_\varepsilon$ for all s in the relevant interval.

An important observation is given momentarily about the versions of the L^2-norm of $w_- - d(\sigma_1 q_{1-})$ on the $r\le\rho_*-\frac{1}{512}\varepsilon$ part of $\mathcal{N}_\varepsilon$. Keep in mind in what follows that this 2-form is zero on the $r>\rho_*-\frac{1}{256}\varepsilon$ part of $\mathcal{N}_\varepsilon$. Given $s\in[-100,-96]$, the notation uses $\|w_- - d(\sigma_1 q_{1-})\|_s$ to denote version of the L^2-norm of $w_- - d(\sigma_1 q_{1-})$ on the $r<\rho_*-\frac{1}{512}\varepsilon$ part of $\mathcal{N}_\varepsilon$. There is the analogous definition for $\|w_+ - d(\sigma_1 q_{1+})\|_s$ when $s\in[96,100]$. Here is the key observation:

(9.40)
- Each $s\in[-100,-96]$ version of $\|w_- - d(\sigma_1 q_{1-})\|_s$ is bounded by c_0 times the L^2-norm of w_- on Y_-.
- Each $s\in[96,100]$ version of $\|w_+ - d(\sigma_1 q_{1+})\|_s$ is bounded by c_0 times the L^2-norm of w_+ on Y_+.

To see about (9.40), write any $s\in[-100,-96]$ or $s\in[96,100]$ version of $\mathfrak{g}_{T_\diamond}$ at any given point in the $r<\rho_*-\frac{1}{512}\varepsilon$ part of $\mathcal{N}_\varepsilon$ as

$$\mathfrak{g}_{T_\diamond} = \lambda_1\hat{e}^1\otimes\hat{e}^1 + \lambda_2\hat{e}^2\otimes\hat{e}^2 + \lambda_3\hat{e}^3\otimes\hat{e}^3 \tag{9.41}$$

with λ_1, λ_2 and λ_3 being positive numbers and with $\{\hat{e}^k\}_{k=1,2,3}$ being a $\mathfrak{g}_T$-orthonormal frame. It follows from (9.7)-(9.9) that each λ_k can be written as $(T/T_\diamond)^2 e_k$ where the numbers e_1, e_2 and e_3 are such that $c_0^{-1}\le e_1, e_2\le c_0$ and $c_0^{-1}\le e_3\le c_0(T/T_\diamond)^2$. It follows from this that the volume form of the metric is less than $c_0(T/T_\diamond)^4$ times that of $\mathfrak{g}_T$. It also follows from this that the square of the $\mathfrak{g}_{T_\diamond}$-norm of $w_- - d(\sigma_1 q_{1-})$ is less than $c_0(T/T_\diamond)^4$ times the square of its $\mathfrak{g}_T$ norm. These last two observations imply that the integrand whose integral gives $\|w_- - d(\sigma_1 q_{1-})\|_s^2$ is no greater than c_0 times the integrand whose integral computes the square the $\mathfrak{g}_T$ version of the L^2-norm of $w_- - d(\sigma_1 q_{1-})$. This last fact implies directly the first bullet of (9.40). But for replacing $-$ subscripts with $+$ subscripts, the same argument proves the second bullet of (9.40).

Part 11: This part of the subsection completes the proof of Proposition 3.10. According to Part 8, it is sufficient to find the closed 2-form p_X with certain special properties. This is done given two more constraints on $\mathfrak{m}$. The first contraint affects $\mathfrak{m}$ only on the $|s| \in [96, 100]$ part of X. The statement of this uses the embeddings from the second and third bullets of (2.7) to view the $s < 0$ and $s > 0$ parts of X as $(-\infty, 0] \times Y_-$ and as $(0, \infty) \times Y_+$:

(9.42) The metric $\mathfrak{m}$ on $[-100, -96] \times \mathcal{Y}_M$ is the product metric $ds^2 + \mathfrak{g}_-$ when $Y_- = Y_0$; and when $Y_+ = Y_0$, the metric $\mathfrak{m}$ on $[96, 100] \times \mathcal{Y}_M$ is the product metric $ds^2 + \mathfrak{g}_+$.

To state the second constraint, re-introduce from Part 7 of Section 9.1 the set Θ and the associated collection $\{\mathcal{T}_\gamma\}_{(\gamma, Z_\gamma)\in\Theta}$ of subsets of M_δ. The following observation views them as subsets of Y_0 and M:

There exists an embedding of $\mathbb{R} \times (\bigcup_{(\gamma,Z_\gamma)\in\Theta} \mathcal{T}_\gamma)$ into X with the following two properties:

(9.43)
- The function s on X pulls back via the embedding to its namesake on the $\mathbb{R}$-factor of $\mathbb{R} \times (\bigcup_{(\gamma,Z_\gamma)\in\Theta} \mathcal{T}_\gamma)$.
- The composition of this embedding of the $|s| > 1$ part of $\mathbb{R} \times (\bigcup_{(\gamma,Z_\gamma)\in\Theta} \mathcal{T}_\gamma)$ with the inverse of the embeddings from the second and third bullets of (2.7) is the tautological inclusion map.

The existence of such an embedding is implied by what is said in the first paragraph of this section about the ascending and descending manifolds from the critical point of s. The second constraint uses $\mathfrak{m}_-$ and $\mathfrak{m}_+$ to denote the metrics $ds^2 + \mathfrak{g}_-$ and $ds^2 + \mathfrak{g}_+$ on the product $\mathbb{R} \times (\bigcup_{(\gamma,Z_\gamma)\in\Theta} \mathcal{T}_\gamma)$.

(9.44) There exists a T-independent constant, $c_* > 1$, with the following significance: The pull-back of $\mathfrak{m}$ via the embedding in (9.43) obeys $c_*^{-1}\mathfrak{m}_- \le \mathfrak{m} \le c_*\mathfrak{m}_-$ and $c_*^{-1}\mathfrak{m}_+ \le \mathfrak{m} \le c_*\mathfrak{m}_+$.

Granted these constraints, the three steps that follow construct p_X when $Y_- = Y_0$. The construction when $Y_+ = Y_0$ is not given as it has the identical description but for changes of $-$ to $+$ in various places.

Step 1: Define p_X on the $s \in [-102, -101]$ part of X to be $p_{\mathcal{N}1}$ and define p_X on the $s \in [-101, -100]$ part of X to be $p_{\mathcal{N}2}$. The rest of this step extends the definition of p_X to the $s \in [-100, -98]$ part of X. To this end, use the embedding from the second bullet of (2.7) to view this part of X as $[-100, -98] \times Y_0$.

The 2-form $p_{\mathcal{N}2}$ near $s = -100$ is the s-independent 2-form on Y_0 given by p_0 on $\mathcal{Y}_0$ and $w_- - d(\sigma_1 q_{1-})$ on the rest of Y_0. This understood, p_X is extended to the $s \in [-100, -98]$ part of X as this s-independent 2-form on Y_0.

Write the metric $\mathfrak{m}$ appearing on $[-100, -98] \times Y_0$ as $ds^2 + \mathfrak{g}$ with $\mathfrak{g}$ denoting an s-dependent metric on Y_0. The constraint in (9.42) asserts that $\mathfrak{g} = \mathfrak{g}_-$ on $\mathcal{Y}_M$. Meanwhile, $\mathfrak{g}$ is Part 9's metric on the $r < \rho_* - \frac{1}{512}\varepsilon$ part of $\mathcal{N}_\varepsilon$. It therefore follows from (9.40) that the L^2-norm of p_X on Y as defined by any $s \in [-100, -98]$ version of $\mathfrak{g}$ is bounded by c_0.

Step 2: This step extends the definition of p_X to the $s \in [-98, -96]$ part of X. To do this, view the $s \in [-98, -96]$ part of X as $[-98, -96] \times Y_0$ as in Step 1. Keep in mind for what follows that the metric $\mathfrak{m}$ here has the form $ds^2 + \mathfrak{g}_M$ with $\mathfrak{g}_M$ being an s-independent metric on Y_0. Note in particular that $\mathfrak{g}_M = \mathfrak{g}_-$ on $\mathcal{Y}_M$ and it is the metric that is depicted in (9.6) on the $r \leq \rho_* - \frac{1}{1024}\varepsilon$ part of $\mathcal{N}_\varepsilon$.

Lemma 9.5 is invoked momentarily to construct a 1-form on the union of $\mathcal{Y}_M$ and the $r < \rho_*$ part of $\mathcal{N}_\varepsilon$, with the three properties listed momentarily. The list of properties denotes the 1-form by q_{3-} and it reintroduces the 2-form p from Part 7 of Section 9.1. Here are the three properties: The 1-form q_{3-} obeys $dq_{3-} = p - w_- + d(\sigma_1 q_{1-})$, it vanishes on the r $\geq \rho_* - \frac{1}{512}\varepsilon$ part of $\mathcal{N}_\varepsilon$, and its L^2-norm as defined by the $\mathfrak{g}_M$ is bounded by c_0 times the L^2-norm of w_- on Y.

Let $\chi_{\diamond 3}$ denote the function on $\mathbb{R}$ given by $\chi(|s| - 97)$. The function $\chi_{\diamond 3}$ equals 0 where $|s| \geq 98$ and it equals 1 where $|s| \leq 97$. Introduce $\chi'_{\diamond 3}$ to denote its derivative. The 2-form p_X on the $s \in [-98, -96]$ part of X is p_0 on $\mathcal{Y}_0$ and it is given on the rest of Y_0 by

$$ds \wedge \chi'_{\diamond 3} q_{3-} + w_- - d(\sigma_1 q_{1-}) + \chi_{\diamond 3}\, dq_{3-}. \tag{9.45}$$

Of particular note is that the $\mathfrak{m}$ version of the L^2-norm of the 2-form p_X on $[-98, -96] \times Y_0$ is bounded by c_0. What follows is a key point to keep in mind for Step 3: The 2-form p_X on $[-97, -96] \times Y_0$ is the 2-form $p_0 + p$ from Y_0.

Lemma 9.5 is invoked using for the set U the union of $\mathcal{Y}_M$ and the $r < \rho_* - \frac{1}{1024}\varepsilon$ part of $\mathcal{N}_\varepsilon$. Lemma 9.5's set V is the union of $\mathcal{Y}_M$ and the $r < \rho_* - \frac{1}{512}\varepsilon$ part of $\mathcal{N}_\varepsilon$. The

2-form $w_{\mathfrak{g}}$ is $p - w_- + d(\sigma_1 q_{1-})$. Note that this 2-form is zero on the $r > \rho_* - \frac{1}{256}\varepsilon$ part of U. The metric used for the lemma is the metric $\mathfrak{g}_M$. It follows from (9.40) and (9.42) that the L^2-norm of $p - w_- + d(\sigma_1 q_{1-})$ as defined by $\mathfrak{g}_M$ is bounded by c_0. As neither U, V nor $\mathfrak{g}_M$ depend on T, Lemma 9.5 finds a 1-form q on U with $dq = p - w_- + d(\sigma_1 q_{1-})$ whose L^2-norm on V is bounded by c_0. To obtain q_{3-} from q, note that q on the $r > \rho_* - \frac{1}{256}\varepsilon$ part of $\mathcal{N}_\varepsilon$ is given by $d\mathfrak{k}$ with $\mathfrak{k}$ denoting a smooth function. Changing $\mathfrak{k}$ by a constant if necessary supplies a version whose L^2-norm is bounded by c_0 times that of $d\mathfrak{k}$; thus by c_0. Take such a version. Meanwhile, use χ to construct a smooth function of r on $\mathcal{N}_\varepsilon$ that equals 0 where $r \geq \rho_* - \frac{1}{512}\varepsilon$, equals 1 where r $\leq \rho_* - \frac{1}{256}\varepsilon$ and whose derivative has norm bounded by c_0. Denote this function by σ_3 and set $q_{3-} = q - d(\sigma_3 \mathfrak{k})$.

Step 3: This step extends the definition of p_X to the $s \in [-96, 102]$ part of X. To this end, consider first the definition of p_X on the $s \in [-96, 100]$ part of X. As p_0 is supported in the image of the embedding from (9.10) and as the 2-form p is supported in the image of the embedding from (9.43), these embeddings can be used to view $p_0 + p$ as a 2-form on the $s \in [-96, 100]$ part of X. View them in this light and define p_X on this same part of X to be $p_0 + p$. The constraint in (9.44) has the following implication: The L^2-norm of p_X on the $s \in [-96, 100]$ part of X is bounded by c_0.

The definition of p_X on the $s \in [100, 102]$ part of X views this part of X via the embedding from the third bullet of (2.7) as $[100, 102] \times (M \sqcup (S^1 \times S^2))$. The 2-form p_0 on $S^1 \times S^2$ can be written as $w + dq_0$ with q_0 being a smooth 1-form. Likewise, the 2-form p on M can be written as $w_+|_M + dq_M$ with q_M denoting a smooth 1-form. Set $q_{4+} = q_0 + q_M$. Let $\chi_{\diamond 4}$ denote the function on $\mathbb{R}$ given by $\chi(s - 100)$. This function $\chi_{\diamond 4}$ is equal to 1 where $s < 100$ and it is equal to 0 where $s > 101$. Use $\chi'_{\diamond 4}$ to denote its derivative.

Define p_X on the $s \in [100, 102]$ part of X to be the 2-form

$$ds \wedge \chi'_{\diamond 4} q_{4+} + p_0 + p - \chi_{\diamond 4} d q_{4+} \tag{9.46}$$

This form is closed, and it extends p_X as a 2-form that equals w_+ where $s > 101$. Of particular note is that the L^2-norm of p_X on the $s \in [100, 102]$ part of X is bounded by c_0. □

9.5 Proof of Proposition 3.12

The proof of this proposition has two parts. Of the two possible cases, only that where $Y_- = Y_k$ and $Y_+ = Y_{k-1} \sqcup (S^1 \times S^2)$ is discussed as the case when the roles are

switched is proved with the same argument but for changing the direction of various inequalities and signs that involve s.

Part 1 of what follows proves the first bullet of Proposition 3.12. The subsequent parts of this subsection address the assertion in the second bullet and in doing so, they define implicitly the required subset $\mathrm{Met}(Y_k)$. To make the definition only slightly less implicit, return momentarily to what is said about *Met* just prior to Part 1 of Section 9.4. By way of a reminder, each metric in *Met* is determined in part by a metric from the Y_0 version of Section 9.1's set $\mathrm{Met}^{\mathcal{N}}$ and a sufficiently large choice for a number denoted by T. A lower bound on T is determined by certain properties of the chosen $\mathrm{Met}^{\mathcal{N}}$ metric. This said, a metric from *Met* is in Proposition 3.12's subset $\mathrm{Met}(Y_k)$ if and only if the chosen value for T is larger than a new lower bound. This new lower bound is determined in part by the same properties of the chosen $\mathrm{Met}^{\mathcal{N}}$ metric that determine the $\mathrm{Met}(Y_0)$ lower bound. The chosen metrics on the $S^1 \times S^2$ components also determine in part the lower bound for T. By the way, no generality is lost by taking the metrics on these components to be the product of the standard Euclidean S^1 and the standard round metric on S^2. In any event, this new lower bound is determined implicitly by the constructions in Parts 2-10.

Part 1: This part discusses the first bullet of the proposition. The notation used below is that used to describe Y and its geometry in [KLT1]-[KLT4]. In particular, the manifold Y and its 2-form w are described in Section II.1. A summary of the salient features can be found in Section IV.1a. The notation used below is the same as that used in Section II.1 and Section IV.1a.

To set the stage, label the G pairs in the set Λ as $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_{\mathrm{G}}\}$. A $k \in \{1, \ldots, \mathrm{G}\}$ version of the manifold Y_k is obtained from Y_0 by attaching k 1-handles, these being the handles from from the set $\{\mathcal{H}_{\mathfrak{p}}\}_{\mathfrak{p} \in \{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}}$. Thus, Y_k is obtained from Y_{k-1} by attaching just the handle $\mathcal{H}_{\mathfrak{p}_k}$. By way of a short review, Y is obtained from Y_0 by a surgery that attaches G 1-handles to $Y_0 - \mathcal{H}_0$. The attaching region of each handle are disjoint coordinate balls centered around a pair of points in $Y_0 - \mathcal{H}_0$. The set of such pairs is denoted by Λ. The 1-handle that corresponds to a given pair $\mathfrak{p} \in \Lambda$ is denoted by $\mathcal{H}_{\mathfrak{p}}$. The geometry of Y_k near $\mathcal{H}_{\mathfrak{p}_k}$ is as follows: The handle $\mathcal{H}_{\mathfrak{p}_k}$ is diffeomorphic to $[-R - 7\ln\delta_*, R + 7\ln\delta_*] \times S^2$ given by the preferred coordinates $(u, (\theta, \phi))$ with u denoting the Euclidean coordinate for interval factor and with (θ, ϕ) denoting spherical coordinates on the constant u cross-sectional spheres of $\mathcal{H}_{\mathfrak{p}_k}$. The handle is attached to Y_{k-1} using the identifications given in (3.3) with it understood that $(r_+, (\theta_+, \phi_+))$ and $(r_-, (\theta_-, \phi_-))$ are certain preferred spherical coordinates for respective balls about the two points that comprise the pair $\mathfrak{p}_k$.

The definition of X requires choosing a properly embedded arc in the $\mathcal{Y}_M$ part of Y_{k-1}. The arc has one end point at one of the points in $\mathfrak{p}_k$ and the other end point at the other. This arc intersects a neighorhood of the boundary of the radius $7\delta_*$ coordinate ball centered at the points from $\mathfrak{p}_k$ as a ray from the origin when viewed using the coordinate system that is specified in Section II.1.a. Part 7 of Section 9.1 introduces a finite set of pairs Θ in M_δ with one partner in each pair being an embedded loop in M_δ. Part 7 of Section 9.1 associates each such loop a small radius tubular neighborhood, this being $\mathcal{T}_\gamma$ when γ is the loop in question. The arc must be chosen so as to lie in the complement of the closure of all such tubular neighborhoods. The arc can and should be chosen to intersect that $f = \frac{3}{2}$ Heegaard surface in M_δ transversely in a single point.

Let $S_{\mathfrak{p}_k} \subset \mathcal{Y}_M$ denote an embedded 2-sphere boundary of neighborhood of the arc with each point having distance between 2δ and 4δ from the arc. The neighborhood in question and S should be disjoint from the closures of the tubular neighborhoods of the loops from Θ. The sphere S appears in Y_k as an embedded 2-sphere that separates Y_k into two components. One of these contains $\mathcal{H}_{\mathfrak{p}_k}$ and is diffeomorphic to the complement in $S^1 \times S^2$ of an embedded ball.

The following is a consequence of what is said above about the descending and ascending submanifolds from the critical points of s: The pseudogradient vector field that defines the embeddings from the second and third bullets of (2.7) can be chosen so that (3.9) are obeyed and likewise (3.10) and the conditions in (9.10) and (9.43). These properties are assumed in what follows. The condition for the first Chern class is satisfied if and only it has zero pairing with the cross-sectional 2-spheres in each $\mathfrak{p} \in \{\mathfrak{p}_1, \ldots, \mathfrak{p}_{k-1}\}$ version of the Y_{k-1} version of $\mathcal{H}_\mathfrak{p}$ and annihilates the generator of $H_2(S^1 \times S^2; \mathbb{Z})$.

Part 2: Proposition 3.12 requires as input a metric from a certain subset of a set of metrics on Y_{k-1} that is denoted by $\mathrm{Met}(Y_{k-1})$ and a metric from a set of metrics on Y_k, this denoted by $\mathrm{Met}(Y_k)$. These subsets of metrics are in the respective Y_{k-1} and Y_k versions of *Met*. They are defined roughly as follows: Let Y_* for the moment denote either Y_{k-1} or Y_k. Each metric in the Y_* version of *Met* is determined in part by a metric from the corresponding version of Met_N, this defined in Section 9.1. The second input for the definition is a large choice for the parameter T. A metric in *Met* of this sort is denoted in Section 9.2 by $\mathfrak{g}_{3T}$. A Y_* metric $\mathfrak{g}_{3T}$ is in $\mathrm{Met}(Y_*)$ if T is greater than a certain lower bound that is determined implicitly by the chosen Met_N metric. As in the case of Proposition 3.12's implicit definition of $\mathrm{Met}(Y_0)$, this lower bound is determined implicitly by the requirements of subsequent constructions. In

any event, it is determined by certain curvature norms, injectivity radius lower bounds and volume.

The construction of a suitable metric on X starts by choosing metrics $\mathfrak{g}_{1-}$ and $\mathfrak{g}_{1+}$ from the respective Y_- and Y_+ versions of Met_N. This done, use what is said in Parts 1-10 of Section 9.4 to define a metric $\mathfrak{m}_{T*}$ and self-dual 2-form ω_{T*} on X. It then follows from what is said in Part 8 and at the start of Part 9 of Section 9.4 that the pair $\mathfrak{m}_{T*}$ and ω_{T*} satisfy the requirements of Proposition 3.12 if there exists a suitable closed 2-form p_X on X with the following properties: The de Rham cohomology class of p_X is that of $c_1(\det(\mathbb{S}))$. In addition, p_X must equal w_- where $s < -102$ and w_+ where $s > 102$ with w_- and w_+ being the respective $\mathfrak{g}_-$ and $\mathfrak{g}_+$ harmonic 2 forms on Y_- and Y_+ with de Rham cohomology class that of $c_1(\det(\mathbb{S}))$.

The construction of p_X is this case differs in only one respect from the construction described in Parts 9-11 of Section 9.4, this involving Step 3 in Part 11 of Section 9.4. To say more about this difference, require as in Part 11 of Section 9.4 that the metric $\mathfrak{m}$ obey (9.42). Require in addition that (9.43) is obeyed; as noted in Part 1 above, such a requirement can be met. With (9.43) understood, the metric $\mathfrak{m}$ is chosen so as to obey the constraints in (9.44). Proceed with the constructions in Steps 1 and 2 of Part 11 in Section 9.4. Step 3 in Part 11 of Section 9.4 is replaced with the following Step 3′:

Step 3′: Define p_X on the $s \in [-96, 96]$ part of X by viewing $p_0 + p$ as a 2-form on this part of X via the embeddings in (9.10) and (9.43). The constraint in (9.44) implies that such a definition yields a version of p with L^2-norm bounded by c_0 on the $s \in [-96, 96]$ part of X. Extend p_X to the $[96, 102]$ part of X by copying almost verbatim what is done in Steps 1 and 2 with the direction of s reversed and with the metric $\mathfrak{g}_+$ in (9.42) used in lieu of $\mathfrak{g}_-$.

9.6 Proof of Proposition 3.13

The construction of the cobordism manifold X, its metric and self-dual 2-form has nine parts.

Part 1: This part sets some of the notation for the construction in the subsequent parts of the subsection of the desired metric on X and the 2-form w_X. Fix a metric on Y of the sort that is described in Part 2 of Section 3.6 and denote the latter by $\mathfrak{g}_Y$. The 2-form w on Y has $\mathfrak{g}_Y$ norm equal to 1 and its Hodge dual is the 1-form $\hat{a}$ that is

described in Section II.3a; see also (IV.1.6). The constant L for use in (2.8) is specified at the end of the proof. Assume until then that $L > 100$ has been chosen.

The description of the metric for X and the 2-form w_X on the $s \in [-L, -L+8]$ part of X requires the formula for w on a given $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_{\mathfrak{p}}$ from (IV.1.3):

$$\begin{aligned} w = {} & 6x\cos\theta\sin\theta d\theta du - \sqrt{6}f'\cos\theta\sin^2\theta\, du\, d\phi \\ & + \sqrt{6}f(1-3\cos^2\theta)\sin\theta\, d\theta\, d\phi. \end{aligned} \tag{9.47}$$

The notation here uses x and f to denote a pair of non-negative functions on $\mathcal{H}_{\mathfrak{p}}$, these given in (IV.1.2), with f' denoting the derivative of f. Both x and f are invariant under the reflection $u \mapsto -u$. The function x has compact support and is a non-zero constant where $|u| < 2$. This constant is denoted by x_0. The function f on the $|u| < 4$ part of $\mathcal{H}_{\mathfrak{p}}$ is given by the rule $u \mapsto f(u) = x_0 + 4e^{-2R}\cosh(2u)$.

The 1-form $\upsilon_\diamond$ given in (IV.1.5) plays a central role in what follows. This 1-form near on the $|u| < 4$ part of $\mathcal{H}_{\mathfrak{p}}$ can be written as

$$\upsilon_\diamond = 4e^{-2R}\cosh(2u)(1-3\cos^2\theta)\, du + 12\, e^{-2R}\sinh(2u)\cos\theta\sin\theta\, d\theta. \tag{9.48}$$

The 1-form $\upsilon_\diamond$ is a closed form on Y, and its zero locus are the loci in each $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_{\mathfrak{p}}$ where both u and the function $1 - 3\cos^2\theta$ are zero. Note also that $*w = \upsilon_\diamond$ on the complement in Y of the $|u| \geq R + \ln\delta - 9$ parts of each $\mathfrak{p} \in \Lambda$ handle $\mathcal{H}_{\mathfrak{p}}$. A second point to note is that $*(w \wedge \upsilon_\diamond) \geq c_0^{-1}|\upsilon_\diamond|^2$ on the whole of Y.

Part 2: Let $*$ denote for the moment the Hodge star of the metric $\mathfrak{g}_Y$ on Y. The desired metric for X must pull back to $(-\infty, -L] \times Y$ via the embedding from the second bullet of (2.7) as the metric $ds^2 + \mathfrak{g}_Y$. Meanwhile, the corresponding pull-back of w_X must equal $ds \wedge *w + w$. This 2-form is self-dual but it is not closed; this is because $d * w \neq 0$ on the $|u| \leq R + \ln\delta - 9$ part of each $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_{\mathfrak{p}}$. This last fact follows from the formula in (IV.1.6).

This rest of this part of the subsection describes w_X for $s \in [-L, -L+3]$. The metric on this part of X still pulls back as $ds^2 + \mathfrak{g}_Y$ via the second bullet of (2.7).

Let $\chi_{\diamond 1}$ denote the function on $\mathbb{R}$ given by the rule $s \mapsto \chi(-s-L+2)$. This function is equal to 0 where $s < -L+1$ and it is equal to 1 where $s > -L+2$. The derivative of $\chi_{\diamond 1}$ is denoted in subsequent equations by $\chi'_{\diamond 1}$. Fix $m > 1$ and introduce χ_m to denote the function of the coordinate s given by the rule $s \mapsto \chi(m|u| - 1)$. This function

equals 0 where $|u| > 2m^{-1}$ and it equals 1 where $|u| < m^{-1}$. By way of a look ahead, m will be set equal to $r^{1/c_0 c}$ when the time comes to verify the requirements of Proposition 3.12.

Use w_1 to denote the s-dependent 2-form on Y that is equal to w on the $M_\delta \cup \mathcal{H}_0$ part of Y, and equal to the following 2-form below on each $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_\mathfrak{p}$:

$$
\begin{aligned}
w_1 = d\big(x(1-\chi_{\diamond 1}\chi_m)\,(1-3\cos^2\theta)\,du\big) - \sqrt{6}f'\cos\theta\sin^2\theta\,du\,d\phi \\
+ \sqrt{6}f\,(1-3\cos^2\theta)\sin\theta\,d\theta\,d\phi.
\end{aligned}
\tag{9.49}
$$

Note that $|w_1| \le c_0$. Meanwhile, $\frac{\partial}{\partial s}w_1 = d\mathfrak{b}$ with $\mathfrak{b} = -x\chi'_{\diamond 1}\chi_m(1-3\cos^2\theta)du$. As $\chi_m = 0$ where $|u| > 2m^{-1}$, the L^2-norm of $\mathfrak{b}$ on $[-L, -L+3]\times Y$ is no greater than $c_0 m^{-1}$. The appearance of $\chi_{\diamond 1}$ in the definition guarantees that $w_1 = w$ where $s \le -L$. Note that w_1 is a closed 2-form on Y for each s. A key point to note is that zero set of the $s > -L+1$ versions of w_1 consists of two circles in each $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_\mathfrak{p}$, these being the circles where u and $1-3\cos^2\theta$ are both zero.

The desired 2-form w_X pulls back to $[-L, -L+3]\times Y$ via the embedding from the second bullet of (2.7) as $ds \wedge *w_1 + w_1$.

Part 3: What follows directly describes the desired metric and the 2-form w_X on the $s \in [-L+3, -L+4]$ part of X. To this end, let $\chi_{\diamond 2}$ denote the function on $\mathbb{R}$ that is given by the rule $s \mapsto \chi(s+L-3)$. This function is equal to 1 where $s \le -L+3$ and it is equal to 0 where $s \ge -L+4$. A smooth metric on Y will be constructed momentarily whose Hodge star sends the $s \ge -L+3$ versions of w_1 to $\upsilon_\diamond$, thus making w_1 harmonic. Let $\mathfrak{g}_1$ denote this metric. Use $\mathfrak{g}$ to denote the s-dependent metric $\chi_{\diamond 2}\mathfrak{g}_Y + (1-\chi_{\diamond 2})\mathfrak{g}_1$ and let $*$ now denote its Hodge dual. The metric on X pulls back $[-L+3, -L+4]\times Y$ via the embedding from the second bullet of (2.7) as $ds^2 + \mathfrak{g}$. The pull back of w_X to $[-L+3, -L+4]\times Y$ is the 2-form $ds \wedge *w_1 + w_1$. This 2-form is self-dual when s is near $-L+4$. The two steps that follow construct the metric $\mathfrak{g}_1$.

Step 1: The 2-form w_1 is equal to w on the $M_\delta \cup \mathcal{H}_0$ part of Y and its $\mathfrak{g}_Y$ Hodge star here is $\upsilon_\diamond$. This understood, the metric $\mathfrak{g}_1$ on $M_\delta \cup \mathcal{H}_0$ is set equal to $\mathfrak{g}_Y$. To define $\mathfrak{g}_1$ on a given $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_\mathfrak{p}$, note first that the function $\chi_{\diamond 1}$ in (9.49) is equal to 1 when $s \in [-L+3, -L+4]$. This implies that w_1 is s-independent when $s \in [-L+3, -L+4]$. More to the point, it also implies that the $s \in [-L+3, -L+4]$ version of w_1 shares the same zero locus with the closed 1-form $\upsilon_\diamond$, this being the

circles in each $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_{\mathfrak{p}}$ where u and $1-3\cos^2\theta$ are both zero. Meanwhile $w_1 \wedge \upsilon_\diamond > 0$ on the complement of their common zero locus. This last observation can be used with Lemma 9.2 to construct the desired metric $\mathfrak{g}_1$ on any part of the complement in $\mathcal{H}_{\mathfrak{p}}$ of the $u = 0$ and $1 - 3\cos^2\theta = 0$ locus as a smooth extension of the metric $\mathfrak{g}_Y$ from $M_\delta \cup \mathcal{H}_0$.

Step 2: Let $\mathcal{T} \subset \mathcal{H}_{\mathfrak{p}}$ denote the $|u| < m^{-1}$ part of $\mathcal{H}_{\mathfrak{p}}$. The function χ_m in (9.36) is equal to 1 on $\mathcal{T}$ and $f = x_0 + 4e^{-2R}\cosh(2u)$ on $\mathcal{T}$. This being the case, it follows from (9.47) and (9.48) that the metric on $\mathcal{T}$ with volume 3-form $\Omega = \sin\theta\, du\, d\theta\, d\phi$ and Hodge star defined by the rules:

$$\left\{\begin{array}{l} *\sin\theta d\theta d\phi = \frac{1}{\sqrt{6}}\frac{4e^{-2R}\cosh(2u)}{x_0+4\mathrm{r}^{-2R}\cosh(2u)}du, \\ *\sin\theta d\phi\, du = \frac{3}{2\sqrt{2}}d\theta, \\ *du\, d\theta = \frac{3}{2\sqrt{2}}\sin\theta\, d\phi \end{array}\right. \tag{9.50}$$

sends w_1 to $\upsilon_\diamond$. Note that a suitable change of coordinates near the $\theta = 0$ and $\theta = \pi$ loci can be used to prove that the metric defined by (9.50) is smooth on the whole of $\mathcal{T}$.

As noted previously, Lemma 9.2 can be used to extend the metric defined in (9.50) to the whole of $\mathcal{H}_{\mathfrak{p}}$ so as to agree with $\mathfrak{g}_Y$ on $\mathcal{H}_{\mathfrak{p}} \cap M_\delta$. This must be done with some care so as to obtain an $m = \mathrm{r}^{1/c_0 c}$ extension that can be used to satisfy the second item of (3.13). With this goal in mind, note that Lemma 9.2 can be used to find an extension with the following three properties:

(9.51)
- The norm of the Riemannian curvature tensor and those of its covariant derivatives to order 20 are bounded by c_0.
- The injectivity radius is bounded from below by c_0^{-1}.
- The metric volume of Y is at most c_0.

The first bullet of Lemma 9.2 gives metrics that obey the third bullet of (9.51) and the second bullet of Lemma 9.2 supplies metrics that obey all three bullets.

Part 4: The desired metric for X and the 2-form w_X on the $s \in [-L+4, -\frac{3}{4}L+2]$ portion of X are described below. This is done by specifying their pull-backs via the embedding from the second bullet of (2.7) to $[-L+4, -\frac{3}{4}L+2] \times Y$. In this part, we use $\chi_{\diamond 2}$ to denote the function on $\mathbb{R}$ given by the rule $s \mapsto \chi\big(\frac{4}{L-20}(s+L-5)\big)$. This

function is equal to 1 where $s < -L + 5$ and it is equal to zero where $s > -\frac{3}{4}L$. Use $\chi'_{\diamond 2}$ to denote the derivative of $\chi_{\diamond 2}$. Note in particular that $|\chi'_{\diamond 2}| \leq c_0 L^{-1}$.

Let w_2 denote the s-dependent 2-form on Y given by w_1 for $s < -L + 4$, given by w on $M_\delta \cup \mathcal{H}_0$, and given on each $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_\mathfrak{p}$ for $s \geq -L + 4$ by

$$
\begin{aligned}
w_2 = {} & \chi_{\diamond 2}\, d\big(x(1 - \chi_m)(1 - 3\cos^2\theta)\, du\big) - \sqrt{6} f' \cos\theta \sin^2\theta\, du\, d\phi \\
& + \sqrt{6} f (1 - 3\cos^2\theta) \sin\theta\, d\theta\, d\phi.
\end{aligned}
\tag{9.52}
$$

The 2-form w_2 is a closed 2-form on Y for each s, it has the same zero locus as w_1 and it has the property that $w_2 \wedge \upsilon_\diamond = w_1 \wedge \upsilon_\diamond$.

An s-dependent metric on Y is described momentarily for the cases when $L > c_0$. This metric is denoted by $\mathfrak{g}$. Let $*$ denote the corresponding Hodge dual. By way of a look ahead, $\mathfrak{g}$ is chosen so that $d * w_2 = \frac{\partial}{\partial s} w_2$. The pull back of the desired metric on X to $[-L + 4, -\frac{3}{4}L + 2] \times Y$ via the embedding from the second bullet of (2.7) is the quadratic form $ds^2 + \mathfrak{g}$, and the corresponding pull back of w_X is $ds \wedge *w_2 + w_2$. Note in particular that w_X is self-dual and closed if self-duality is defined by the metric $ds^2 + \mathfrak{g}$.

The metric $\mathfrak{g}_1$ from Part 3 is s-independent and so it is defined where $s > -L + 4$. This understood, the metric $\mathfrak{g}$ is set equal to $\mathfrak{g}_1$ where $s < -L + 5$. It is also set equal to $\mathfrak{g}_1$ for all $s \in [-L + 4, -\frac{3}{4}L + 2]$ on $M_\delta \cup \mathcal{H}_0$. This is to say that it equals $\mathfrak{g}_Y$ for all such s on $M_\delta \cup \mathcal{H}_0$. The metric $\mathfrak{g}$ is chosen where $s \geq -L + 5$ on each $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_\mathfrak{p}$ so that its Hodge star on each $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_\mathfrak{p}$ acts on w_2 as

$$
* w_2 = \chi'_{\diamond 2} x (1 - \chi_m)(1 - 3\cos^2\theta)\, du + \upsilon_\diamond.
\tag{9.53}
$$

As will be explained directly, if $L > c_0$, there are metrics of the sort just described that obey the $c_0 = 1$ version of (9.51) where $s > -\frac{3}{4}L + 1$.

To see about these requirements, consider first constructing a metric of the desired sort where $s > -\frac{3}{4}L$. The metric that is defined by (9.50) with volume form $\sin\theta\, du\, d\theta\, d\phi$ satisfies the requirements where $|u| < 2$. Since $w_2 \wedge \upsilon_\diamond > 0$ on the rest of $\mathcal{H}_\mathfrak{p}$ and the $\mathfrak{g}_Y$ Hodge star of w_2 is $\upsilon_\diamond$ on $M_\delta \cup \mathcal{H}_0$, Lemma 9.2 finds an extension of the latter metric from the $|u| < 1$ part of each $\mathcal{H}_\mathfrak{p}$ that has the desired properties. Use $\mathfrak{g}_2$ to denote this s-independent metric.

Consider next the story where $s < -\frac{3}{4}L + 1$. The metric on any given $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_\mathfrak{p}$ that is defined by (9.50) with volume form $\sin\theta\, du\, d\theta\, d\phi$ has Hodge star sending w_2 to $\upsilon_\diamond$ where $|u| < m^{-1}$. Let υ denote the 1-form on the right hand side of (9.53). The 3-form $\upsilon \wedge w_2$ can be written where $|u| \geq \frac{1}{2} m^{-1}$ as $\mathfrak{q} \upsilon_\diamond \wedge w_2$ and it follows from the

fact that $|\chi'_{\diamond 2}| < c_0 L^{-1}$ that $\mathfrak{q} > c_0^{-1} - c_0 L^{-1}$. Thus, $\upsilon \wedge w_2 > 0$ where $|u| > \frac{1}{2} m^{-1}$. Given this positivity and given what was said in the preceding paragraphs, Lemmas 9.2 and 9.3 can be used to construct an s-dependent metric where $s < -\frac{3}{4}L + 1$ that equals $\mathfrak{g}_2$ where $s > -\frac{3}{4}L + \frac{1}{2}$, that equals $\mathfrak{g}_1$ where $s < -L + 5$ and equals $\mathfrak{g}_Y$ on $M_\delta \cup \mathcal{H}_0$.

Part 5: This part and Part 6 construct the desired metric for X and the 2-form w_X where $s \in [-\frac{3}{4}L + 1, -\frac{1}{2}L + 2]$. By way of a look ahead, the metric pulls back from this part of X via the embedding from the second bullet of (2.7) as $ds^2 + \mathfrak{g}_3$ with $\mathfrak{g}_3$ being an s-dependent metric on Y that equals the metric $\mathfrak{g}_*$ for all s on the set $\mathcal{Y}_{0\varepsilon}$ from (9.10).

The metric $\mathfrak{g}_3$ is independent of s on the whole of Y when $s \in [-\frac{1}{2}L + 1, -\frac{1}{2}L + 2]$. This s-independent version of $\mathfrak{g}_3$ is in a large T version of the space Met_T that is defined in Part 5 of Section 9.1. For the purposes to come, the choice of T requires choosing $L > c_T$ with c_T denoting here and in what follows a constant that depends on T and is greater than $c_0 T^2$ in any event. The value of c_T may increase between appearances.

Use $*$ now to denote the $\mathfrak{g}_3$ Hodge star on Y. The 2-form w_X pulls back via the embedding from the second bullet of (2.7) to $[-\frac{3}{4}L + 1, -\frac{1}{2}L + 2] \times Y$ as $ds \wedge *w_3 + w_3$, with w_3 denoting an s-dependent, closed 2-form on Y. The 2-form w_3 is also independent of s where $s \in [-\frac{1}{2}L + 1, -\frac{1}{2}L + 2]$ and it is independent of s on $\mathcal{Y}_{0\varepsilon}$ for all s. With regards to the motivation for what follows below and in Part 6, keep in mind that $ds \wedge *w_3 + w_3$ is closed if and only if both $dw_3 = 0$ and $d(*w_3) = \frac{\partial}{\partial s} w_3$ for all s.

This part of the subsection makes the assumption that $c_1(\det(\mathbb{S}))$ annihilates the $H_2(M;\mathbb{Z})$ summand of the direct sum decomposition for $H_2(Y;\mathbb{Z})$ given in (IV.1.4). This assumption makes for a simpler construction. Even so, much of what is done here is used again for Part 6's construction for the general case.

The construction that follows has six steps. Note that some of these steps use notation from Section 9.1.

Step 1: Let $\chi_{\diamond 3}$ denote the function of s given by $\chi\big(\frac{3}{L-8}(s + \frac{3}{4}L - 2)\big)$. This function equals 1 for $s < -\frac{3}{4}L + 2$ and it equals 0 for $s \geq -\frac{1}{2}L$. Reintroduce the notation from Section 9.1 and let χ_r denote the function on $\mathbb{R}^3$ given by $\chi(64\,\varepsilon_*^{-1}(r - \rho_*) - 1)$. This

function equals 1 where $r < \rho_* + \frac{1}{64}\varepsilon$ and it equals 0 where $r > \rho_* + \frac{1}{32}\varepsilon$. Given $T \geq 1$ and use χ_r with $\chi_{\diamond 3}$ to define the s-dependent function on $\mathbb{R}^3$ given by

$$r_{s_T} = \chi_{\diamond 3} r + (1 - \chi_{\diamond 3})\big(1 - \chi_r + \frac{1}{T}\chi_r\big)r. \tag{9.54}$$

Note in particular that $\frac{\partial}{\partial s} r_{s_T} > 0$ because χ_r is a non-increasing function of r. Use ρ_{s_T} and $x_{s_T 3}$ to denote the respective s-dependent functions on $\mathbb{R}^3$ given by $r_{s_T}\sin\theta$ and $r_{s_T}\cos\theta$.

Define the s-dependent 2-form w_3 on Y by setting $w_3 = w_2$ for $s \leq -\frac{3}{4}L + 2$ and setting it equal to w on the $\mathcal{Y}_0$ component of $Y - \mathcal{N}_\varepsilon$. The 2-form w_3 is defined on $\mathcal{N}_\varepsilon$ by specifying it on the $\mathbb{R}^3$ incarnation of $\mathcal{N}_\varepsilon$ to be $\kappa(\rho_{s_T})\,\rho_{s_T}\,d\rho_{s_T}\,d\phi$. The definition of w_3 on the rest of Y uses τ to denote the function of s given by $(\chi_{\diamond 3} + (1 - \chi_{\diamond 3})/T)^2$. The latter function equals 1 where $s < -\frac{3}{4}L + 2$ and it is equal to $\frac{1}{T^2}$ where $s > -\frac{1}{2}L$. The 2-form w_3 is defined on $\mathcal{Y}_M \cap M_\delta$ to be τw_2; and it is defined on each $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_\mathfrak{p}$ in the upcoming (9.55). This upcoming definition uses χ_Δ to denote the function of u and θ given by $\chi(|u|^2 - 1)\,\chi(4(1 - 3\cos^2\theta) - 1)$. The function χ_Δ is equal to 1 where both $|u| < 1$ and $|1 - 3\cos^2\theta| < \frac{1}{4}$; and it is equal to 0 where either $|u| > 2$ or $|1 - 3\cos^2\theta| > \frac{1}{2}$. Note in particular that the support of χ_Δ consists of two open sets. These are mirror images under the involution $\theta \mapsto \pi - \theta$, with one being a neighborhood of the $u = 0$ and $\cos\theta = \frac{1}{\sqrt{3}}$ circle with $0 < \theta < \frac{\pi}{2}$ on its closure. Define

$$w_3 = -\sqrt{6}\tau\, d\big(f\cos\theta\sin^2\theta d\phi - (x_0 + 4e^{-2R})\,\mathrm{sign}(\cos\theta)\chi_\Delta d\phi\big) \tag{9.55}$$

on $\mathcal{H}_\mathfrak{p}$ for $s > \frac{3}{4}L + 2$.

By way of comparison, the 2-form w_2 on $\mathcal{H}_\mathfrak{p}$ can be written as $\sqrt{6}\,d(f\cos\theta\sin^2\theta d\phi)$. What is written in (9.55) adds a 2-form with support on $\mathcal{H}_\mathfrak{p}$ to τw_2.

The 2-form w_3 on Y is closed for each s. Moreover, it defines the s-independent de Rham cohomology class $c_1(\det(\mathbb{S}))$ because the latter class is assumed to annihilate the $H_2(M;\mathbb{Z})$ summand in (IV.1.4).

Step 2: The s-dependent metric $\mathfrak{g}_3$ is defined when $s \in [-\frac{3}{4}L + 1, -\frac{1}{2}L + 2]$ with the help of a certain s-dependent 1-form, $\mathfrak{b}$. The 1-form $\mathfrak{b}$ should obey $d\mathfrak{b} = \frac{\partial}{\partial s} w_3$. There are four additional constraints on $\mathfrak{b}$. The first is that $\mathfrak{b}$ should vanish on $\mathcal{Y}_0$ and on the part of $\mathcal{N}_\varepsilon$ where $r > \rho_* + \frac{1}{16}\varepsilon$. The second constraint specifies $\mathfrak{b}$ on the $|u| < 4$ part of $\mathcal{H}_\mathfrak{p}$:

$$\mathfrak{b} = -\sqrt{6}\tau'\big(f\cos\theta\sin^2\theta - (x_0 + 4e^{-2R})\,\mathrm{sign}(\cos\theta)\chi_\Delta\big)d\phi, \tag{9.56}$$

where τ' denotes $\frac{\partial}{\partial s}\tau$. The third constraint asks that $\mathfrak{b}$'s norm at $s \in [-\frac{3}{4}L+1, -\frac{1}{2}L+2]$ when measured by the metric $\mathfrak{g}_Y$ obeys $|\mathfrak{b}|_{\mathfrak{g}_-} \leq c_T L^{-1}$. The fourth contraint requires the following: Fix $k \in \{0, \cdots, 20\}$. Then the $\mathfrak{g}_Y$-covariant derivatives up to order 20 of $(\frac{\partial}{\partial s})^k \mathfrak{b}$ are bounded by $c_T L^{-k-1}$.

To see about satifying these constraints, note first that $\mathfrak{b}$ can be chosen to vanish on $\mathcal{Y}_0$ and on the $r > \rho_* + \frac{1}{16}\varepsilon$ part of $\mathcal{N}_\varepsilon$ because w_3 is constant on these parts of Y, and because the first cohomology of the $r \in [\rho_* + \frac{1}{32}\varepsilon, \rho_* + \frac{1}{64}\varepsilon]$ part of $\mathcal{N}_\varepsilon$ is zero. The $c_0 L^{-1}$ bound on $|\chi'_{\diamond 3}|$ implies that $\mathfrak{b}$ can be chosen to vanish on $\mathcal{Y}_0$ and so that its norm elsewhere when measured by the metric $\mathfrak{g}_Y$ is bounded by $c_0 L^{-1}$. A 1-form of this sort can be chosen so that the $\mathfrak{g}_Y$ norms of its derivatives also have the required norm bound. Let $\mathfrak{b}_*$ denote such a choice, and let $\mathfrak{b}_\Lambda$ denote the 1-form on any given $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_\mathfrak{p}$ given by (9.47). Their difference, $\mathfrak{b}_* - \mathfrak{b}_\Lambda$, is a closed 1-form on $\mathcal{H}_\mathfrak{p}$. As $H^1(\mathcal{H}_\mathfrak{p} \cap M_\delta; \mathbb{R}) = 0$, this difference can be written as $d\mathfrak{k}$ with $\mathfrak{k}$ denoting a function on $\mathcal{H}_\mathfrak{p}$. The function $\mathfrak{k}$ can be taken so that $|\mathfrak{k}| \leq c_0 L^{-1}$since the $\mathfrak{g}_Y$ norms of both $\mathfrak{b}_*$ and $\mathfrak{b}_\Lambda$ obey a similar $c_0 L^{-1}$ bound. Granted this bound on $\mathfrak{k}$, then $\mathfrak{b} = \mathfrak{b}_* - d(\chi(|u| - 4)\mathfrak{k})$ has all of the requisite properties.

Step 3: The definition of the upcoming Steps 4 and 6 use observations made below about w_3 and $\mathfrak{b}$ on the $|u| \leq 4$ part of each $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_\mathfrak{p}$. The first series of observations concern w_3. To start, note that the zero locus of the 2-form in (9.56) is the same as that of $\upsilon_\diamond$, this being the locus where both $u = 0$ and $1 - 3\cos^2\theta = 0$. The reason being that f' and χ_Δ have the same sign where $\chi_\Delta \neq 0$, and likewise the functions $(1 - 3\cos^2\theta)$ and $\operatorname{sign}(\cos\theta)\chi_\Delta$ have the same sign where $\chi_\Delta \neq 0$. In fact, these comments about the derivatives of χ_Δ imply that w_3 on $\mathcal{H}_\mathfrak{p}$ can be written schematically as

$$\begin{aligned} w_3 = &-(1 + \mathrm{A}_1)\tau\sqrt{6}f'\cos\theta\sin^2\theta\, du\, d\phi \\ &+ (1 + \mathrm{A}_2)\tau\sqrt{6}f(1 - 3\cos^2\theta)\sin\theta\, d\theta\, d\phi \end{aligned} \tag{9.57}$$

where A_1 and A_2 are smooth, non-negative functions of u and θ that equal zero where both $|u| < 1$ and $|1 - 3\cos^2\theta| < \frac{1}{4}$ and where either $|u| > 2$ or $|1 - 3\cos^2\theta| > \frac{1}{2}$. Given that w_2 on $\mathcal{H}_\mathfrak{p}$ is $-\sqrt{6}\, d(f\cos\theta\sin^2\theta d\phi)$, these last remarks imply that

$$w_3 \wedge \upsilon_\diamond \geq \tau w_2 \wedge \upsilon_\diamond \quad \text{on } \mathcal{H}_\mathfrak{p} \tag{9.58}$$

with the inequality being a strict one only where $d\chi_\Delta \neq 0$.

The next series of remarks concern the 1-form $\mathfrak{b}$ on the $|u| \leq 4$ part of $\mathcal{H}_\mathfrak{p}$. The first point of note being that $f(u)\cos\theta\sin^2\theta$ is equal to $(x_0 + 4e^{-2R})\frac{2}{3\sqrt{3}}\operatorname{sign}(\cos\theta)$ on the

zero locus of $\upsilon_\diamond$. It follows as a consequence that $\flat$ can be written as

$$\flat = -\mathrm{B}_1\tau' f' \cos\theta \sin^2\theta d\phi + \mathrm{B}_2\tau' f(1-3\cos^2\theta)\sin\theta d\phi, \tag{9.59}$$

where B_1 and B_2 are smooth functions of u and θ.

Step 4: The metric $\mathfrak{g}_3$ on each $\mathfrak{p}\in\Lambda$ version of $\mathcal{H}_\mathfrak{p}$ is defined to be the metric from Part 5 for $s < -\frac{3}{4}L+2$. The metric $\mathfrak{g}_3$ on $\mathcal{H}_\mathfrak{p}$ at other values of s is defined in part so that its Hodge star obeys

$$* w_3 = \tau\upsilon_\diamond + \flat. \tag{9.60}$$

There is one other constraint. To explain it, note first that the metric $\mathfrak{g}_2$ does not depend on s when if $s\in[-\frac{3}{4}L+1, -\frac{3}{4}L+2]$. Use $\mathfrak{g}_{2+}$ to denote this s-independent metric. Look at (9.45) to see that the $s > -\frac{1}{2}L+1$ version of w_3 on the $|u|>4$ part of each $\mathcal{H}_\mathfrak{p}$ is $\frac{1}{T^2}w_2$. Since $\flat$ is zero when $s > -\frac{1}{2}L+1$, the constraint in (9.60) is satisfied by taking the Hodge star to be that defined by $\mathfrak{g}_{2+}$. This understood, the final constraint is as follows:

(9.61) The metric $\mathfrak{g}_3$ on each $\mathfrak{p}\in\Lambda$ version of $\mathcal{H}_\mathfrak{p}$ when $s > -\dfrac{1}{2}L+1$ must be both s-independent and T-independent; and it must equal $\mathfrak{g}_{2+}$ where $|u|>4$.

As explained in what follows, an s-dependent metric with all of these requisite properties exists if L is greater than a T-dependent constant.

Consider first the existence of a metric with the desired properties where $|u|<1$ and $|1-3\cos^2\theta| < \frac{1}{4}$, this being a neighborhood of the common zero locus of w_3 and $\upsilon_\diamond$. The metric $\mathfrak{g}$ is defined on this part of $\mathcal{H}_\mathfrak{p}$ by its volume 3-form $\Omega = \sin\theta\, du\, d\theta\, d\phi$ and the Hodge duals

$$\begin{cases} *\sin\theta\, d\theta\, d\phi = \frac{1}{\sqrt{6}}\frac{4e^{-2R}\cosh(2u)}{x_0+4e^{-2R}\cosh(2u)}du + \tau^{-1}\tau'\mathrm{B}_2\sin\theta\, d\phi, \\ *\sin\theta\, d\phi\, du = \frac{\sqrt{3}}{2\sqrt{2}}d\theta - \frac{1}{\sqrt{6}}\tau^{-1}\tau'\mathrm{B}_1\sin\theta\, d\phi, \\ *du\, d\theta = \frac{\sqrt{3}}{2\sqrt{2}}\sin\theta\, d\phi + \frac{1}{\sqrt{6}}\tau^{-1}\tau'\mathrm{B}_2 du - \frac{1}{\sqrt{6}}\tau^{-1}\tau'\mathrm{B}_1 d\theta. \end{cases} \tag{9.62}$$

These formulas for the Hodge dual define a symmetric, bilinear form on the cotangent bundle of this part of $\mathcal{H}_\mathfrak{p}$. This bilinear form is positive definite if $\tau^{-1}|\tau'| < c_0^{-1}$, which is guaranteed if $T^2L^{-1} < c_0^{-1}$ since $\tau^{-1} < T^2$ and $|\tau'| < c_0 L^{-1}$.

To see about defining $\mathfrak{g}_3$ on the rest of $\mathcal{H}_\mathfrak{p}$, use the fact that $|\flat| \le c_0L^{-1}$ to draw the following conclusion: If $L > c_0T^2$, then $w_3\wedge(\tau\upsilon_\diamond+\flat) > 0$ on the complement in Y

of the $|u| < \frac{1}{2}$ and $|1 - 3\cos^2\theta| < \frac{1}{8}$ part of each $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_{\mathfrak{p}}$. This being the case, then Lemma 9.3 can be used directly to obtain a family of metrics on $\mathcal{H}_{\mathfrak{p}}$ parametrized by the set $[-\frac{3}{4}L + 1, -\frac{1}{2}L + 2]$ so as to obey (9.60) and (9.61). Use $\mathfrak{g}_{3\Lambda}$ to denote this family of metrics on $\bigcup_{\mathfrak{p}\in\Lambda} \mathcal{H}_{\mathfrak{p}}$.

Step 5: The 1-form $\upsilon_\diamond$ is used here to construct another closed, s-dependent 1-form that plays a central role in the upcoming definition of the $s \in [-\frac{3}{4}L + 1, -\frac{1}{2}L + 2]$ versions of $\mathfrak{g}_3$ on $M_\delta \cup \mathcal{H}_0$. This new 1-form is denoted by $\upsilon_{\diamond 3}$ and its definition is given in the subsequent paragraph.

The 1-form $\upsilon_{\diamond 3}$ on $\mathcal{Y}_0$ is $\upsilon_\diamond$ and it is defined on the $r > \rho_* - \frac{1}{4}\varepsilon$ part of $\mathcal{N}_\varepsilon$ to be $dx_{s_T 3}$ with the latter defined in Step 1. Since $\upsilon_\diamond = dx_3$ on $\mathcal{N}_\varepsilon$, it follows from the definition of $x_{s_T 3}$ that $\upsilon_{\diamond 3}$ as defined so far is a 1-form on the union of $\mathcal{Y}_0$ and the $r > \rho_* - \frac{1}{4}\varepsilon$ part of $\mathcal{N}_\varepsilon$. The definition of $\upsilon_{\diamond 3}$ on the $r \in [\rho_* - \frac{1}{2}\varepsilon, \rho_* - \frac{1}{4}\varepsilon]$ part of $\mathcal{N}_\varepsilon$ requires the reintroduction of the function χ_{r*} from Step 2 in Part 5 of Section 9.1. This function is used here to define $x_{s_T 3*} = \big(\chi_{\diamond 3} + (1 - \chi_{\diamond 3})(1 - \chi_{r*} + \frac{1}{T}\chi_{\mathrm{r}*})\big)\, x_3$. Define $\upsilon_{\diamond 3}$ on the $r \in [\rho_* - \frac{1}{2}\varepsilon, \rho_* - \frac{1}{4}\varepsilon]$ part of $\mathcal{N}_\varepsilon$ to be $\tau^{1/2} dx_{s_T 3*}$. It follows from the definitions of $x_{s_T 3}$ and $x_{s_T 3*}$ that the definition just given defines a smooth 1-form on the union of $\mathcal{Y}_0$ with the $r > \rho_* - \frac{1}{2}\varepsilon$ part of $\mathcal{N}_\varepsilon$. As the latter's restriction near the $r = \rho_* - \frac{1}{2}\varepsilon$ is τdx_3, a smooth 1-form on $\mathcal{Y}_0 \cup \mathcal{N}_\varepsilon$ is defined by setting $\upsilon_{\diamond 3} = \tau dx_3$ on the $r \leq \rho_* - \frac{1}{2}\varepsilon$ part of $\mathcal{N}_\varepsilon$. Noting that $\tau dx_3 = \tau \upsilon_\diamond$, defining $\upsilon_{\diamond 3}$ on $\mathcal{Y}_M$ to be $\tau\upsilon_\diamond$ defines a smooth, closed 1-form on Y.

The 1-form $\upsilon_{\diamond 3}$ has the four properties that are listed below.

PROPERTY 1: *The 1-form $\upsilon_{\diamond 3}$ is equal to $\upsilon_\diamond$ where $s \in [-\frac{3}{4}L + 1, -\frac{3}{4}L + 2]$.*

This follows because $\chi_{\diamond 3} = 1$ at these values of s.

PROPERTY 2: *The zero locus of each $s \in [-\frac{3}{4}L + 1, -\frac{1}{2}L + 2]$ version of $\upsilon_{\diamond 3}$ is identical to that of $\upsilon_\diamond$.*

This is because $\upsilon_{\diamond 3}$ has no zeros on $\mathcal{Y}_0 \cup \mathcal{N}_\varepsilon$ and it is equal to $\tau\upsilon_\diamond$ on $\mathcal{Y}_M$.

PROPERTY 3: *Each $s \in [-\frac{3}{4}L + 1, -\frac{1}{2}L + 2]$ version of $w_3 \wedge \upsilon_{\diamond 3}$ is positive on the complement of the common zero locus of w_3 and $\upsilon_{\diamond 3}$.*

This property follows directly from the definitions on $Y - (\bigcup_{\mathfrak{p}\in\Lambda} \mathcal{H}_{\mathfrak{p}})$ and from (9.57) on each $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_{\mathfrak{p}}$.

To set the stage for the fourth property, note that w_3 and $\upsilon_{\diamond 3}$ do not depend on s when $s \in [-\frac{3}{4}L+1, -\frac{3}{4}L+2]$. Use w_{3+} and $\upsilon_{\diamond 3+}$ to denote these s-independent differential forms. To continute the stage setting, let $\mathfrak{g}_{3\Lambda_+}$ denote the s-independent metric on $\bigcup_{\mathfrak{p}\in\Lambda}\mathcal{H}_{\mathfrak{p}}$ given by the $s \in [-\frac{3}{4}L+1, -\frac{3}{4}L+2]$ version of Part 5's metric $\mathfrak{g}_{3\Lambda}$. What with (9.51), this metric on $\bigcup_{\mathfrak{p}\in\Lambda}\mathcal{H}_{\mathfrak{p}}$ with $\mathfrak{g}_{2+}$ on $Y - \{\bigcup_{\mathfrak{p}\in\Lambda}\mathcal{H}_{\mathfrak{p}}\}$ define a smooth, s- and T- independent metric on Y. Denote the latter by $\mathfrak{g}_\diamond$. The restriction of $\mathfrak{g}_\diamond$ to $\mathcal{Y}_M \cup \mathcal{N}_\varepsilon$ is in the space $\mathrm{Met}^{\mathcal{N}}$ from Part 5 of Section 9.1. This understood, let $\mathfrak{g}_{\diamond T}$ denote the Met_T metric that is constructed in Part 5 of Section 9.1 from T and $\mathcal{Y}_M\cup\mathcal{N}_\varepsilon$ part of $\mathfrak{g}_\diamond$.

PROPERTY 4: *The $\mathfrak{g}_{\diamond T}$ Hodge star of w_{3+} is $\upsilon_{\diamond 3+}$.*

The definitions in Part 5 of Section 9.1 with those given above for w_{3+} and $\upsilon_{\diamond 3+}$ imply this on $Y - (\bigcup_{\mathfrak{p}\in\Lambda}\mathcal{H}_{\mathfrak{p}})$ and (9.60), (9.61) imply this on $\bigcup_{\mathfrak{p}\in\Lambda}\mathcal{H}_{\mathfrak{p}}$.

Step 6: This step completes the definition of $\mathfrak{g}_3$ on Y so as to satisfy five constraints, the first being that $*w_3 = \upsilon_{\diamond 3} + \mathfrak{b}$ at each $s \in [-\frac{3}{4}L+1, -\frac{1}{2}L+2]$. The second contraint asks that the $s \in [-\frac{3}{4}L+1, -\frac{3}{4}L+2]$ versions are independent of s; and the third asks that the $s \in [-\frac{1}{2}L+1, -\frac{1}{2}L+2]$ versions are also independent of s and that this s-independent metric is $\mathfrak{g}_{\diamond T}$. The fourth constraint asks that $\mathfrak{g}_3 = \mathfrak{g}_{3\Lambda}$ on the $|u| < R + \ln\delta$ part of each $\mathfrak{p}\in\Lambda$ version of $\mathcal{H}_{\mathfrak{p}}$. The fifth and final constraint asks that $\mathfrak{g}_3 = \mathfrak{g}_*$ on $\mathcal{Y}_0$ and on the $r > \rho_* + \frac{1}{16}\varepsilon$ part of $\mathcal{N}_\varepsilon$.

Use PROPERTY 3 and what is said in Step 4 with the bound $|\mathfrak{b}|_{\mathfrak{g}_-} < c_0L^{-1}$ to see that $w_3 \wedge (\upsilon_{\diamond 3} + \mathfrak{b}) > 0$ on the complement in Y of the common zeros of w_3 and $\upsilon_{\diamond 3}$ if $L \geq c_T$. Given this bound, Lemma 9.3 with the input from Step 4 and PROPERTY 4 of Step 5 find a metric with all of the desired properties. Take such a metric for $\mathfrak{g}_3$. Note for future reference that the s-independent, $s > -\frac{1}{2}L+1$ version of $\mathfrak{g}_3$ is equal to $\mathfrak{g}_Y$ on $\mathcal{Y}_M \cap M_\delta$.

Part 6: This part of the subsection puts no constraints on the restriction of $c_1(\det(\mathbb{S}))$ to the $H_2(M;\mathbb{Z})$ summand in $H_2(Y;\mathbb{Z})$. The s-dependent metric $\mathfrak{g}_3$ and the 2-form w_3 in this case are identical to their namesakes in Part 5 on $Y - (\bigcup_{(\gamma,Z_\gamma)\in\Theta}\mathcal{T}_\gamma)$. The three steps that follow define $\mathfrak{g}_3$ and w_3 on $\bigcup_{(\gamma,Z_\gamma)\in\Theta}\mathcal{T}_\gamma$.

Step 1: Reintroduce from Part 7 of Section 9.1 the closed 2-form p on Y. By way of a reminder, the de Rham class of p has pairing 0 with the $H_2(\mathcal{H}_0;\mathbb{Z}) \oplus (\bigoplus_{\mathfrak{p}\in\Lambda}\mathcal{H}_{\mathfrak{p}})$

summand in (IV.1.4)'s decomposition of $H_2(Y;\mathbb{Z})$ and its pairing with the $H_2(M;\mathbb{Z})$ summand is the same as that of $c_1(\det(\mathbb{S}))$. Since p's support lies in $\bigcup_{(\gamma,\mathrm{z}_\gamma)\in\Theta}\mathcal{T}_\gamma$ and thus in $\mathcal{Y}_M - (\bigcup_{\mathfrak{p}\in\Lambda}\mathcal{H}_\mathfrak{p})$, setting w_3 on $\mathcal{Y}_M \cup (\bigcup_{\mathfrak{p}\in\Lambda}\mathcal{H}_\mathfrak{p})$ to be $w_3 = \tau w_2 + (1-\tau)p$ defines a closed 2-form on Y for each $s \in [-\frac{3}{4}L+1, -\frac{1}{2}L+2]$ with de Rham cohomology class $c_1(\det(\mathbb{S}))$.

The metric $\mathfrak{g}_3$ is defined on $\bigcup_{(\gamma,\mathrm{z}_\gamma)\in\Theta}\mathcal{T}_\gamma$ so that its Hodge star maps w_3 to $\tau\upsilon_\diamond + \mathfrak{b}$ with $\mathfrak{b}$ denoting a certain 1-form with $d\mathfrak{b} = \frac{\partial}{\partial s}w_3$. As done previously, Lemma 9.3 will be used to construct a metric with this property that meets all of the other requirements.

Step 2: The definition of $\mathfrak{g}_3$ and $\mathfrak{b}$ on $\bigcup_{(\gamma,\mathrm{z}_\gamma)\in\Theta}\mathcal{T}_\gamma$ requires what is said here about the w_2 and p in the support of p. To start, reintroduce from Part 7 of Section 9.1 the set Θ and write p as $\sum_{(\gamma,\mathrm{z}_\gamma)\in\Theta}\mathrm{z}_\gamma p_\gamma$ with each $(\gamma,\mathrm{z}_\gamma)$ version of p_γ being a closed 2-form with support in the tubular neighborhood $\mathcal{T}_\gamma$ that is described in Part 7 in Section 9.1. Part 7 of Section 9.1 describes a diffeomorphism from $S^1\times D$ to $\mathcal{T}_\gamma$ with D denoting a small radius disk about the origin in $\mathbb{R}^2$. The diffeomorphism identifies γ with $S^1\times\{0\}$ and it has two important property that concern the 2-form w on Y and the function f from Section II.1. As noted in Part 7 of Section 9.1, the 1-form df pulls back via the embedding of $S^1\times D$ as a constant 1-form on the D factor and the kernel of the pull back via the embedding of the 2-form w is a constant vector field that is tangent to this D factor. These last properties are exploited in the next paragraph.

As can be seen in (IV.1.5), the 1-form $\upsilon_\diamond$ on $\mathcal{T}_\gamma$ is df. Meanwhile, the 2-form w_2 on $\mathcal{T}_\gamma$ is still the original 2-form w on Y as described in (IV.1.3). This understood, what was said above about df and the kernel of w imply that $S^1\times D$ has coordinates $(t,(x,y))$ with t denoting an affine coordinate for the S^1 factor and (x,y) coordinates for D with the following two properties: The 1-form $\upsilon_\diamond$ pulls back as dx and the 2-form w_2 pulls back as $\mathrm{H}_\gamma(y,t)\,dy\,dt$ with H_γ denoting a positive function. Granted these coordinates, the 2-form p_γ has the form $\mathfrak{h}_\gamma(x,y)\,dx\,dy$ with $\mathfrak{h}_\gamma$ denoting a function with compact support in a small radius disk about the origin in the (x,y)-plane and with total integral equal to 1.

Step 3: An almost verbatim repeat of what is said in Step 2 of Part 6 supplies a version of the 1-form $\mathfrak{b}$ which obeys the four properties listed in the first paragraph of Step 2 in Part 6 with it understood that w_3 is now defined as in Step 1.

It follows as a consequence of what is said in Step 2 that

$$(\tau w_2 + (1-\tau)p)\wedge\upsilon_\diamond = \tau w_2\wedge\upsilon_2, \tag{9.63}$$

and thus the $\mathfrak{g}_Y$-norm of $(\tau w_2 + (1-\tau)p) \wedge (\tau \upsilon_\diamond + \mathfrak{b})$ is no less than $\tau^2(c_0^{-1} - c_T T^2 L^{-1})$. This being the case, Lemma 9.3 supplies an s-dependent metric on Y with all of the desired properties if L is larger than a purely T-dependent constant.

Let $\mathfrak{g}_{3+}$ denote the s-independent metric on Y given by the $s \in [-\frac{1}{2}L + 1, -\frac{1}{2}L + 2]$ versions of $\mathfrak{g}_3$. This is the metric $\mathfrak{g}_Y$ on $(\mathcal{Y}_M \cap M_\delta) - (\bigcup_{(\gamma, \mathrm{Z}_\gamma) \in \Theta} \mathcal{T}_\gamma)$. It proves necessary for what follows to take some care with regards to the choice of $\mathfrak{g}_{3+}$ on $\bigcup_{(\gamma, \mathrm{Z}_\gamma) \in \Theta} \mathcal{T}_\gamma$. In particular, Lemmas 9.2 and 9.3 will construct a version of $\mathfrak{g}_3$ with $\mathfrak{g}_{3+}$ on each $\mathcal{T}_\gamma$ by $\mathfrak{g}_Y$-volume 3-form $\mathrm{H}_\gamma dx\, dy\, dt$ and the Hodge star rules:

$$
(9.64) \qquad \begin{cases} *dx\, dy = \mathrm{A}_\gamma dt - \mathrm{A}_\gamma \tau^{-1}(1-\tau)\, \mathrm{H}_\gamma^{-1} \mathrm{Z}_\gamma \mathfrak{h}_\gamma dx + \mathrm{B}_\gamma dy, \\ *dy\, dt = \mathrm{H}_\gamma^{-1}(1 + \tau^{-2} \mathrm{H}_\gamma^{-1} \mathrm{A}_\gamma (1-\tau)^2 \mathrm{Z}_\gamma \mathfrak{h}_\gamma) dx \\ \qquad\qquad - \mathrm{A}_\gamma \tau^{-1}(1-\tau) \mathrm{H}_\gamma^{-1} \mathrm{Z}_\gamma \mathfrak{h}_\gamma dt, \\ *dt\, dx = dy + \mathrm{B}_\gamma dt, \end{cases}
$$

with A_γ being a positive function and with τ equal to $\frac{1}{T^2}$. The function A_γ is constrained for the moment only to the extent that $\mathrm{A}_\gamma < c_0^{-1} \tau^2$ on the support of $\mathrm{Z}_\gamma \mathfrak{h}_\gamma$ and that A_γ is independent of T on the complement in $\mathcal{T}_\gamma$ of a T-independent open set that contains the support of $\mathfrak{h}_\gamma$ and has compact closure in $\mathcal{T}_\gamma$. This set is denoted by $\mathcal{T}'_\gamma$. This upper bound on A_γ is needed so that (9.64) defines a positive definite metric. As for B_γ, it is zero on $\mathcal{T}_\gamma$ and it it is independent of T elsewhere.

Part 7: This part of the subsection defines the desired metric on X and 2-form w_X on the $s \in [-\frac{1}{2}L + 1, -\frac{1}{2}L + 5]$ part of X. As done previously, these are defined by their pull-backs via the embedding from the second bullet of (2.7). The pull-back of the metric will have the form $ds^2 + \mathfrak{g}$ with $\mathfrak{g}$ denoting an s-dependent metric on Y. Meanwhile, the pull-back of w_X will have the form $ds \wedge *w_4 + w_4$, with w_4 denoting a closed, s-dependent 2-form on Y and with $*$ denoting the Hodge $*$ defined by $\mathfrak{g}$. The de Rham cohomology class of w_4 at each s is $c_1(\det(\mathbb{S}))$.

The metric $\mathfrak{g}$ is independent of s for $s \in [-\frac{1}{2}L + 1, -\frac{1}{2}L + 2]$ and the 2-form w_4 is independent of s for $s \in [-\frac{1}{2}L + 1, -\frac{1}{2}L + 3]$. Both the metric and w_4 are independent of s when $s \in [-\frac{1}{2}L + 4, -\frac{1}{2}L + 5]$. Moreover, the restriction of both to $Y - (\bigcup_{\mathfrak{p} \in \Lambda} \mathcal{H}_\mathfrak{p})$ are independent of s for all values of s. The salient difference between the $s \le -\frac{1}{2}L + 3$ version of w_4 and the $s \ge -\frac{1}{2}L + 4$ version being that the latter has nondegenerate zeros and the former does not.

The construction of $\mathfrak{g}$ and w_4 has two steps.

Step 1: Let $\mathfrak{g}_{3+}$ denote the $-\frac{1}{2}L+2$ version of the metric that is supplied in Parts 5 and 6, and let w_{3+} denote the $s=-\frac{1}{2}L+2$ version of w_3. The 2-form w_{3+} is $\mathfrak{g}_{3+}$-harmonic but it does not vanish transversely. By way of a reminder, the zero locus of w_{3+} consists of the two circles in each $\mathfrak{p}\in\Lambda$ version of $\mathcal{H}_{\mathfrak{p}}$ where both $u=0$ and $1-3\cos^2\theta=0$. Note in this regard that w_{3+} on $\mathcal{H}_{\mathfrak{p}}$ is the 2-form

$$\sqrt{6}T^{-2}(-f'\cos\theta\sin^2\theta\,du\,d\phi+f(1-3\cos^2\theta)\sin\theta\,d\theta\,d\phi). \tag{9.65}$$

The construction of w_4 starts by introducing $\chi_{\diamond 4}$ to denote the function on $\mathbb{R}$ given by $s\mapsto\chi(s+L-3)$. This function is equal to 1 where $s<-\frac{1}{2}L+3$ and it is equal to 0 where $s>-\frac{1}{2}L+3$. The derivative of $\chi_{\diamond 3}$ is denoted by $\chi'_{\diamond 3}$. Use χ_* to denote the function of u given by the rule $u\mapsto\chi(|u|-1)$. This function is equal to 1 where $|u|\le 1$ and it is equal to 0 where $|u|>2$. One last function is needed for what follows, this denoted by χ_θ. It is a function on $[0,\pi]$ with values in $[0,2]$ which has the following two properties: It is zero near the endpoints, and has two local minima at the two values of θ where $1-3\cos^2\theta=0$. Moreover, χ_θ should appear on a neighborhood of these minima as $1+(1-3\cos^2\theta)^2$. Take χ_θ so that $\chi_\theta(\theta)=\chi_\theta(\pi-\theta)$.

Fix $z>1$ and define the 2-form w_z by

$$\begin{aligned} w_z = &-\big(\sqrt{6}f'\cos\theta\sin^2\theta+z^{-1}\cos\phi\,\chi_{\diamond 4}\,\chi_*\sin\theta\frac{\partial}{\partial\theta}\chi_\theta\big)\,du\,d\phi \\ &+\sqrt{6}f(1-3\cos^2\theta)\sin\theta\,d\theta\,d\phi-z^{-1}\sin\phi\,\chi_{\diamond 4}\,\chi_*\frac{\partial}{\partial\theta}\big(\sin\theta\frac{\partial}{\partial\theta}\chi_\theta\big)\,du\,d\theta. \end{aligned} \tag{9.66}$$

This is a closed 2-form for all s that equals w_{3+} for $s\le-\frac{1}{2}L+3$ and for all s where $|u|>2$. This 2-form is independent of s when $s\ge-\frac{1}{2}L+4$. Moreover, if $z>c_0$, then the s-independent version of w_z defined where $s\ge-\frac{1}{2}L+4$ has a nondegenerate zero locus, this being the four points where $\sin\phi=0$, $1-3\cos^2\theta=0$ and $u=0$.

The desired 2-form w_4 is defined to be w_{3+} on $Y-(\bigcup_{\mathfrak{p}\in\Lambda}\mathcal{H}_{\mathfrak{p}})$ and it is defined on each $\mathfrak{p}\in\Lambda$ version of $\mathcal{H}_{\mathfrak{p}}$ to be a $z>c_0$ version of $T^{-2}w_z$.

Step 2: This step defines the metric $\mathfrak{g}$. This is done by first constructing $\mathfrak{g}$ near the zero locus of w_4 in each $\mathfrak{p}\in\Lambda$ version of $\mathcal{H}_{\mathfrak{p}}$ and then extending the result to the rest of Y with the help of Lemma 9.3.

Fix $z > c_0$ so that w_z as defined in (9.66) has nondegenerate zeros. The 2-form w_z can be written as $d\flat_z$ where $\flat_z$ is given by

$$\begin{aligned}\frac{\sqrt{3}}{2\sqrt{2}}z^{-1}\cos\phi\,\chi_{\diamond 4}\,(\chi_*'\chi_\theta du+\chi_*\chi_\theta' d\theta)-\frac{\sqrt{3}}{2\sqrt{2}}z^{-1}\sin\phi\,\chi_{\diamond 4}\chi_*\chi_\theta d\phi \\ +z^{-1}\sin\phi\,\chi_{\diamond 4}'\,\chi_*\sin\theta\frac{\partial}{\partial\theta}\chi_\theta du.\end{aligned}\tag{9.67}$$

Granted this formula, then $\upsilon + \flat_z$ has the same zero locus as w_z if $z > c_0$, and it also vanishes transversely. Moreover, $w_z \wedge (\upsilon_\diamond + \flat_z)$ can be written as $\mathrm{Q}\sin\theta\, du\, d\theta\, d\phi$ and a calculation finds that $\mathrm{Q} \geq 0$ with equality only on the joint zero locus of w_z and $\upsilon_\diamond + \flat_z$. In fact, the calculation finds $\mathrm{Q} \geq c_0^{-1}(|u|^2 + (1-3\cos^2\theta)^2 + z^{-2}\sin^2\phi\sin^2\theta)$ if $z > c_0$.

With z large and w_4 defined by (9.66) on $\mathcal{H}_\mathfrak{p}$, the metric $\mathfrak{g}$ is defined near the zeros of (9.67) so that its Hodge star sends w_z to $\upsilon_\diamond + \flat_z$. The definition requires the introduction of yet another function of s, this denoted by $\chi_{\diamond\diamond 4}$ and defined by the rule whereby $\chi_{\diamond\diamond 4}(s) = \chi(s + \frac{1}{2}L - 2)$. This function equals 1 where $s < -\frac{1}{2}L + 2$ and it equals 0 where $s > -\frac{1}{2}L + 3$. The desired metric $\mathfrak{g}$ is defined by taking its volume form to be $\sin\theta\, du\, d\theta\, d\phi$ and its Hodge star to act as follows:

$$\left\{\begin{array}{l} *\sin\theta\, d\theta\, d\phi = \frac{1}{\sqrt{6}}\big(4e^{-2R}\cosh(2u) + 12\,z^{-1}\sin\phi\,\chi_{\diamond 4}'\cos\theta\sin^2\theta\big)du. \\ *\sin\theta\, d\phi\, du = \frac{\sqrt{3}}{2\sqrt{2}}d\theta. \\ *du\, d\theta = \frac{\sqrt{3}}{2\sqrt{2}}\big(\chi_{\diamond\diamond 4}\sin\theta + (1-\chi_{\diamond\diamond 4})\,\chi_\theta\,(\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}\chi_\theta))^{-1}\big)d\phi. \end{array}\right.\tag{9.68}$$

By way of a parenthetical remark, the metric $\mathfrak{g}_{3+}$ near the zeros of w_z is defined by the same volume form but with Hodge star rule given by (9.50). The appearance of $\chi_{\diamond\diamond 4}$ in the third line of (9.68) guarantees that $\mathfrak{g} = \mathfrak{g}_{3+}$ where $s \leq -\frac{1}{2}L$.

As noted previously, $w_z \wedge (\upsilon_\diamond + \flat_z) > 0$ on the complement of the common zero locus of w_z and $(\upsilon_\diamond + \flat_z)$. Having constructed $\mathfrak{g}$ on a neighborhood of this locus with the desired properties, Lemma 9.3 provides an extension to the whole of Y which is independent of s where $s < -\frac{1}{2}L+2$, where $s > -\frac{1}{2}L+4$ and on $Y - (\bigcup_{\mathfrak{p}\in\Lambda}\mathcal{H}_\mathfrak{p})$. This extension is such that the 2-form $ds \wedge *w_4 + w_4$ is self-dual on $[-\frac{1}{2}L+1, -\frac{1}{2}L+5] \times Y$ when self duality is defined by the metric $ds^2 + \mathfrak{g}$.

Part 8: This part of the subsection supplies the input for the definition in Part 9 of the desired metric and the 2-form w_X on the $s \in [-\frac{1}{2}L+4, L]$ part of X. The discussion in this section refers to an auxillary copy of the space X, this denoted by X_*. The manifold X_* is the same as X, but its metric is not a metric of the sort that is described

in Parts 1-7. The eight steps that follow construct a metric on X_* and a corresponding self dual 2-form with certain desirable properties.

Step 1: Fix a metric in the Y_{G} version of $\mathrm{Met}^{\mathcal{N}}$. The latter with a sufficiently large choice for T determines metrics in the set $\mathrm{Met}(Y_{\mathrm{G}})$. This understood, choose T large enough so that this is the case and so that two additional requirements are met, the first being that Part 7's metric $\mathfrak{g}$ and 2-form w_4 can be constructed for any choice of $L > c_T$ with c_T denoting a constant that is greater than 1 and depends only on T. The second requirement is given in Step 2.

Let $\mathfrak{g}_-$ and w_- denote the respective $s \in [-\frac{1}{2}L + 4, -\frac{1}{2}L + 5]$ versions of $\mathfrak{g}$ and w_4, these being independent of s. The metric $\mathfrak{g}_-$ is in Y's version of the space Met_T, so it can be used for the metric $\mathfrak{g}_1$ in Part 1 of Section 9.2, and since w_- has non-degenerate zeros, it can also be used for the metric $\mathfrak{g}_2$ in Part 1 of Section 9.2. This part of Section 9.2 uses w_2 to denote the $\mathfrak{g}_2$-harmonic 2-form with de Rham cohomology class that of $c_1(\det(\mathbb{S}))$. This 2-form w_2 is w_-. The 2-form w_- is equal to w on $\mathcal{Y}_0$ and on the $r \geq \rho_* + \frac{5}{8}\varepsilon$ part of $\mathcal{N}_\varepsilon$ and so it follows that w_- is also the 2-form that is denoted by w_3 in Part 2 of Section 9.2. This fact implies that the metric $\mathfrak{g}_-$ is also a version of what Part 2 of Section 9.2 denotes as $\mathfrak{g}_{3T}$. Parts 1-10 of Section 9.4 will be invoked in the upcoming steps using X_* and the $\mathfrak{g}_-$ version of $\mathfrak{g}_{3T}$. These parts of Section 9.4 denote the latter version of $\mathfrak{g}_{3T}$ by $\mathfrak{g}_{-T}$. What Parts 1-10 of Section 9.4 denote as w_{-T} in this case is the 2-form w_-.

Step 2: Let $\mathfrak{g}_\diamond$ denote the given metric from the $\mathrm{Met}(Y_{\mathrm{G}})$. By way of a reminder, the metric $\mathfrak{g}_\diamond$ is determined in part by Step 1's chosen metric from the Y_{G} version of $\mathrm{Met}^{\mathcal{N}}$ and T.

As explained in Part 1 of Section 9.2, a metric denoted by $\mathfrak{g}_2$ determines various versions of the metric $\mathfrak{g}_{3T}$; and $\mathfrak{g}_\diamond$ can be any one of these $\mathfrak{g}_{3T}$ metrics. Set $\mathfrak{g}_+$ to be the version of $\mathfrak{g}_2$ that is used to construct $\mathfrak{g}_\diamond$ and set $\mathfrak{g}_{+T}$ to denote $\mathfrak{g}_\diamond$. What follows is the second requirement for T: It should be large enough so that the $Y_- = Y$ and $Y_+ = Y_{\mathrm{G}}$ version of the constructions in Parts 1-10 from Section 9.4 can be invoked using X_* and the metrics $\mathfrak{g}_-$ on Y_- and $\mathfrak{g}_+$ on Y_+.

The constructions in Parts 1-8 of Section 9.4 require a closed 2-form on X_*, this denoted by p_X, whose de Rham cohomology class is $c_1(\det(\mathbb{S}))$ which has the following additional properties: It equals w_- where $s < -102$, it equals w_+ where $s > 102$ and it obeys the bound in (9.20). Given such a 2-form, Parts 1-8 of Section 9.4 supply

$L_1 \gg 1$, a metric on X_*, and a 2-form on X_* with the properties listed below. The metric and 2-form are denoted in the list and subsequently by $\mathfrak{m}_{T*}$ and ω_{T*}.

(9.69)
- The metric $\mathfrak{m}_{T*}$ obeys (2.8) and (3.12) when the version of L in the latter is greater than $L_1 + 20$.
- The pull-back of $\mathfrak{m}_{T*}$ from the $s < -L_1 - 1$ part of X via the embedding from the second bullet of (2.7) is $ds^2 + \mathfrak{g}_-$ and the pull back of $\mathfrak{m}_{T*}$ from the $s > L_1 + 1$ part of X_* by the embedding from the third bullet of (2.7) is $ds^2 + \mathfrak{g}_+$.
- The 2-form ω_{T*} is self dual when self duality is defined by $\mathfrak{m}_{T*}$. In addition, the pull-back of ω_{T*} to any constant, $s > 1$ slice of X_* is closed.
- The pull back of ω_{T*} from the $s < -L_1 - 1$ part of X_* by the embedding from the second bullet of (2.7) is $ds \wedge *w_- + w_-$ with $*$ denoting here the $\mathfrak{g}_-$-Hodge star.
- The pull back of ω_{T*} from the $s > L_1 + 1$ part of X_* via the embedding from the third bullet of (2.7) is $ds \wedge *w_+ + w_+$ with $*$ now denoting the $\mathfrak{g}_+$-Hodge star and with w_+ denoting the $\mathfrak{g}_+$-harmonic 2-form with de Rham cohomology class $c_1(\det(\mathbb{S}))$.
- The 2-form ω_{T*} obeys the constraint in (3.11).
- The norm of ω_{T*} and those of its $\mathfrak{m}_{T*}$-covariant derivatives to order 10 are less than c_0.

When comparing the notation in (9.69) with the notation in Parts 1-10 of Section 9.4, keep in mind that this case has $\mathfrak{g}_{-T} = \mathfrak{g}_-$ and $w_{-T} = w_-$, and $\mathfrak{g}_{+T} = \mathfrak{g}_\diamond$ and $w_{+T} = w_\diamond$.

The remaining steps construct a version of p_X with the required properties.

Step 3: The construction of p_X requires the three constraints on $\mathfrak{m}_{T*}$ that are described here and a fourth constraint that is described in Step 4. The first constraint is that imposed in Part 10 of Section 9.4.

The remaining constraints and that in Step 4 refer to the subset $\bigcup_{(\gamma, Z_\gamma)\in\Theta} \mathcal{T}_\gamma \subset M_\delta$, this viewed as a subset of Y and also as a subset of Y_G. The second constraint uses the embeddings from the first and second bullets of (2.7) to view the $s < 0$ and $s > 0$

parts of X_* as $(-\infty, 0] \times Y$ and as $(0, \infty) \times Y_G$. This constraint is the analog of that given in (9.42).

$$\begin{aligned} &\text{The metric } \mathfrak{m}_{T*} \text{ on } [-100, -96] \times \mathcal{Y}_M \text{ is the product metric } ds^2 + \mathfrak{g}_Y. \\ &\text{The metric } \mathfrak{m}_{T*} \text{ on } [96, 100] \times \mathcal{Y}_M \text{ is the product metric } ds^2 + \mathfrak{g}_+. \end{aligned} \tag{9.70}$$

By way of background for the third constraint, note that (9.43) holds for X_*, this being a consequence of what is said in Part 1 about the ascending and descending manifolds from the critical points of s. The third constraint refers to this embedding. It also uses $\mathfrak{m}_Y$ and $\mathfrak{m}_+$ to denote the metrics $ds^2 + \mathfrak{g}_Y$ and $ds^2 + \mathfrak{g}_+$ on $\mathbb{R} \times \bigcup_{(\gamma, \mathrm{Z}_\gamma) \in \Theta} \mathcal{T}_\gamma$.

$$\begin{aligned} &\text{There exists a } T\text{-independent constant, } c_* > 1, \text{ with} \\ &\text{the following significance: The pull-back of } \mathfrak{m}_{T*} \text{ from} \\ &\text{the } s > -94 \text{ part of } X_* \text{ via the embedding in (9.43) obeys} \\ &c_*^{-1}\mathfrak{m}_Y \le \mathfrak{m} \le c_*\mathfrak{m}_Y \text{ and } c_*^{-1}\mathfrak{m}_+ \le \mathfrak{m} \le c_*\mathfrak{m}_+. \end{aligned} \tag{9.71}$$

This third constraint is the analog of the constraint in (9.44).

Step 4: This step describes the fourth constraint on $\mathfrak{m}_{T*}$. This constraint on $\mathfrak{m}_{T*}$ specifies its pull back to $[-96, -94] \times \bigcup_{(\gamma, \mathrm{Z}_\gamma) \in \Theta} \mathcal{T}_\gamma$ via the embedding from the second bullet of (2.7). The constraint asks that this pull-back have the form $ds^2 + \mathfrak{g}$ with $\mathfrak{g}$ denoting a certain s-dependent metric on $\bigcup_{(\gamma, \mathrm{Z}_\gamma) \in \Theta} \mathcal{T}_\gamma$. The upcoming description of $\mathfrak{g}$ refers to the depiction in (9.64) of $\mathfrak{g}_-$ on $\bigcup_{(\gamma, \mathrm{Z}_\gamma) \in \Theta} \mathcal{T}_\gamma$; and it refers to an analogous depiction of the metric $\mathfrak{g}_Y$ on $\bigcup_{(\gamma, \mathrm{Z}_\gamma) \in \Theta} \mathcal{T}_\gamma$. The metric $\mathfrak{g}_Y$ on each $\mathcal{T}_\gamma$ has the same form as (9.64) but with $\mathfrak{h}_\gamma = 0$ and with different versions of A_γ and B_γ. The $\mathfrak{g}_Y$ versions of these functions are denoted by A_{Y_γ} and B_{Y_γ}. Note that $\mathrm{A}_{Y_\gamma} \ge c_0^{-1}$.

The specification of $\mathfrak{g}$ uses two functions on $\mathbb{R}$, the first being the function $\chi_{\diamond 1}^{\mathcal{T}}$ given by $\chi(s + 96)$. This function equals 1 where $s < -96$ and it equals 0 where $s \ge -95$. The second function is denoted by $\chi_{\diamond 2}^{\mathcal{T}}$, it is given by $\chi(s + 95)$. The latter is equal to 1 where $s < -95$ and it is equal to 0 where $s > -94$.

The metric $\mathfrak{g}$ on $\mathcal{T}_\gamma$ is defined by its volume form, this being $\mathrm{H}_\gamma dxdydt$, and by the following Hodge star rules:

$$\begin{aligned} &* dx\, dy = \\ &\big(\chi_{\diamond 2}^{\mathcal{T}}\mathrm{A}_\gamma + (1 - \chi_{\diamond 2}^{\mathcal{T}})\mathrm{A}_{Y_\gamma}\big)dt - \chi_{\diamond 1}^{\mathcal{T}}\mathrm{A}_\gamma\tau^{-1}(1-\tau)\,\mathrm{H}_\gamma^{-1}\mathrm{Z}_\gamma\mathfrak{h}_\gamma dx + \mathrm{B}_\gamma\, dy, \\ &* dy\, dt = \\ &\mathrm{H}_\gamma^{-1}(1 + \chi_{\diamond 1}^{\mathcal{T}}\tau^{-2}\mathrm{H}_\gamma^{-1}\mathrm{A}_\gamma(1-\tau)^2\mathrm{Z}_\gamma\mathfrak{h}_\gamma)\, dx - \chi_{\diamond 1}^{\mathcal{T}}\mathrm{A}_\gamma\tau^{-1}(1-\tau)\,\mathrm{H}_\gamma^{-1}\mathrm{Z}_\gamma\mathfrak{h}_\gamma dt, \\ &* dt\, dx = \\ &dy + \mathrm{B}_\gamma dt. \end{aligned} \tag{9.72}$$

Important points to note are that $\mathfrak{g}$ is independent of T and s on a neighborhood of $s = -94$, that $\mathfrak{g} = \mathfrak{g}_-$ on a neighborhood of $s = -96$ and that $\mathfrak{g} = \mathfrak{g}_-$ for all s on the complement of $\mathcal{T}'_\gamma$.

Step 5: This step describes p_X and says more about the metric $\mathfrak{m}_{T*}$. The 2-form p_X and the metric $\mathfrak{m}_{T*}$ on the $s \in [-102, -98]$ part of X_* are described by the analog of Step 1 in Part 11 of Section 9.4 that has Y replacing Y_G. By way of a summary, p_X is defined on the $s \in [-102, -101]$ part of X to be the 2-form $p_{\mathcal{N}1}$ that is described in Y's version of Step 3 from Part 9 of Section 9.4. The 2-form p_X is defined on the $s \in [-101, -100]$ part of X to be Y's version of the 2-form $p_{\mathcal{N}2}$ that is described in Step 4 from Part 9 of Section 9.4. The definition of p_X on the $s \in [-100, -98]$ part of X is made by specifying its pull back via the embedding from the second bullet of (2.7). This pull back is the s-independent 2-form that equals p_0 on $\mathcal{Y}_0$ and $w_- - d(\sigma_1 q_{1-})$ on the rest of Y. The metric $\mathfrak{m}_{T*}$ on this part of X pulls back via the embedding from the second bullet of (2.7) as $ds^2 + \mathfrak{g}$ with $\mathfrak{g}$ denoting the metric given by $\mathfrak{g}_-$ on $\mathcal{Y}_M$, the metric in (9.39) on $[-100, -98] \times \mathcal{N}_\varepsilon$ and the metric $\mathfrak{g}_*$ on $[-100, -98] \times \mathcal{Y}_0$. Note in this regard that $\mathfrak{m}_{T*}$ is in any event described by (9.10).

Step 6: This step describes p_X and the metric on the $s \in [-98, -96]$ part of X. But for one significant difference, the description of p_X here is similar to the description of its namesake given in Step 2 from Part 11 in Section 9.4. Both p_X and the metric on this part of X are described by their pull-backs via the embedding from the second bullet of (2.7). The metric pulls back as $ds^2 + \mathfrak{g}$ with $\mathfrak{g}$ given by $\mathfrak{g}_*$ on $\mathcal{Y}_{0\varepsilon}$ and by the metric in (9.39) on $\mathcal{N}_\varepsilon$. The metric $\mathfrak{g}$ on $\mathcal{Y}_M$ is the metric $\mathfrak{g}_-$.

As in the Step 2 from Part 11 of Section 9.4, a 1-form to be denoted by q_{3-} is constructed with the following properties: It obeys $dq_{3-} = p - w_- + d(\sigma_1 q_{1-})$, it vanishes on the $\mathrm{r} \geq \rho_* - \frac{1}{2}\varepsilon$ part of $\mathcal{N}_\varepsilon$ and its L^2-norm is bounded by c_0. Reintroduce $\chi_{\diamond 3}$ to denote the function on $\mathbb{R}$ given by $\chi(|s| - 97)$ and use $\chi'_{\diamond 3}$ to denote its derivative. The 2-form p_X on $[-98, -96] \times Y$ is p_0 on $\mathcal{Y}_0$ and given on the rest of Y by the formula in (9.45). Note that p_X is $p_0 + p$ near $\{-96\} \times Y$, and that its L^2-norm on this part of X is bounded by c_0.

To start the description of q_{3-}, let γ denote a loop from a pair in the set Θ. The 2-form w_- on $\mathcal{T}_\gamma$ is given by $\tau w + (1 - \tau)\mathrm{z}_\gamma p_\gamma$ and so it can be written as

$$p_\gamma + \tau(\mathrm{Q}_\gamma dt - \mathrm{z}_\gamma \mathfrak{q}_\gamma dx), \tag{9.73}$$

where Q_γ is a function of y and t whose y-derivative is H_γ, and where $\mathfrak{q}_\gamma$ is a function of x and y whose y-derivative is $\mathfrak{h}_\gamma$. Meanwhile, $\tau = \frac{1}{T^2}$. Let q_γ denote

$\tau(\mathrm{Q}_\gamma dt - \mathrm{Z}_\gamma \mathfrak{q}_\gamma dx)$. Use (9.64) to see that the $q_\gamma \wedge *q_\gamma$ can be written as $\sigma \mathrm{H}_\gamma dx\,dy\,dt$ with $|\sigma| \le c_0\tau^2 \mathrm{A}_\gamma^{-1}$. Now, A_γ is constrained to be positive and less than $c_0^{-1}\tau^2$, and these constraints are met if A_γ is chosen greater than $c_0^{-2}\tau^2$. Take A_γ so that this is the case, and then the L^2-norm (and pointwise norm) of q_γ is bounded by c_0.

The 2-form $w_- - d(\sigma q_{1-})$ is exact on $\mathcal{Y}_M - (\bigcup_{(\gamma,\mathrm{Z}_\gamma)\in\Theta} \mathcal{T}_\gamma')$ and on the $r \le \rho_*$ part of $\mathcal{N}_\varepsilon$. This being the case, it can be written as dq_* on this part of Y. More to the point, Lemma 9.5 can be used as in the last paragraph of Step 2 from Part 11 in Section 9.4 to obtain a version of q_* that is zero where $\mathrm{r} \ge \rho_* - \frac{1}{512}\varepsilon$ and has L^2-norm bounded by c_0 on $\mathcal{Y}_M - (\bigcup_{(\gamma,\mathrm{Z}_\gamma)\in\Theta} \mathcal{T}_\gamma')$ and on the $\mathrm{r} \le \rho_* - \frac{1}{512}\varepsilon$ part of $\mathcal{N}_\varepsilon$.

Let γ again denote a loop from a pair in Θ. The difference $q_* - q_\gamma$ on $\mathcal{T}_\gamma - \mathcal{T}_\gamma'$ is exact. This being the case, it follows from the Mayer-Vietoris exact sequence and from the fact that the various loops from Θ freely generate $H_1(M_\delta;\mathbb{R})$ that there is a closed 1-form, $\mathcal{k}$, on $\mathcal{Y}_M$ with the following three properties: First, $q_* - q_\gamma = \mathcal{k}$ on each $(\gamma,\mathrm{Z}_\gamma)\in\Theta$ version of $\mathcal{T}_\gamma - \mathcal{T}_\gamma'$. Second, $\mathcal{k} = 0$ near $\mathcal{N}_\varepsilon$ and on $\bigcup_{\mathfrak{p}\in\Lambda}\mathcal{H}_\mathfrak{p}$. Finally, the L^2-norm of $\mathcal{k}$ is bounded by c_0. This understood, the sought after 1-form q_{3-} is defined to be q_γ on each $(\gamma,\mathrm{Z}_\gamma)\in\Theta$ version of $\mathcal{T}_\gamma$ and to be $q_* - \mathcal{k}$ on $\mathcal{Y}_M - \bigcup_{(\gamma,\mathrm{Z}_\gamma)\in\Theta}\mathcal{T}_\gamma$.

Step 7: This step describes p_X and the metric on the $s\in[-96,-94]$ part of X. The story with p_X is simple: It is the 2-form $p_0 + p$. The metric on X is described by its pull back to $[-96,-94]\times Y$ via the embedding from the second bullet of (2.7). In particular, it pulls back as $ds^2 + \mathfrak{g}$ with $\mathfrak{g}$ being an s-dependent metric on Y. The s-dependence involves only $\mathfrak{g}$'s restriction to $\bigcup_{(\gamma,\mathrm{Z}_\gamma)\in\Theta}\mathcal{T}_\gamma$ where it is given in Step 4. The metric $\mathfrak{g}$ is independent of s on the rest of Y. As explained in the next paragraph, this metric on X is such that the L^2-norm of p_X on the $[-96,-94]$ part of X is bounded by c_0, a T-independent constant.

The afore-mentioned L^2-norm bound holds for p_0. To see about p, write it as $\sum_{(\gamma,\mathrm{Z}_\gamma)\in\Theta}\mathrm{Z}_\gamma p_\gamma$. A given version of p_γ has support in $\mathcal{T}_\gamma$ where the metric is given by (9.72). Fix $s\in[-96,-94]$ and since $p_\gamma = h_\gamma dx\,dy$, the first bullet of (9.72) can be used to write $p_\gamma \wedge *p_\gamma$ as $\mathrm{P}|\mathfrak{h}_\gamma|^2\mathrm{H}_\gamma dx\,dy\,dt$ with $\mathrm{P} = (\chi^{\mathcal{T}}_{\diamond 2}\mathrm{A}_\gamma + (1-\chi^{\mathcal{T}}_{\diamond 2})\mathrm{A}_{Y_\gamma})^2$. Since $\mathrm{P} < c_0$, so the L^2-norm of p_γ at any $s\in[-96,-94]$ slice of $[-96,-94]\times Y$ is bounded by c_0.

Step 8: This last step describes p_X and the metric on the $s\in[-94,102]$ part of X. The description of p_X starts where $s\in[96,102]$. The 2-form p_X here is described by the $Y_+ = Y_\mathrm{G}$ version of the 2-form that is defined in Steps 1 and 2 from Part 11 of Section 9.4. The $s\in[96,100]$ part of the constraint in (9.70) and the constraint from

Part 10 of Section 9.4 are needed to repeat Steps 1 and 2 from Part 11 in the case at hand. These steps define a version of p_X whose L^2-norm on the $s \in [96, 102]$ part of X is bounded by c_0 times the L^2 norm of w_+ on Y_{G}. This version of p_X is w_+ near the $s = 102$ slice of X_* and it is the 2-form $p_0 + p$ near the $s = 96$ slice. The 2-form p_X is set equal to $p_0 + p$ on the $s \in [-94, 96]$ part of X. Its L^2-norm on the $s \in [-94, 96]$ part of X is bounded by c_0, this being a consequence of (9.71).

Part 9: Taking up where Part 8 left off, this last part of the subsection defines the desired metric on X and w_X on the $s \in [-\frac{1}{2}L+4, L]$ part of X. To this end, fix T large and then $L_1 \geq c_T$ so as to use the constructions in Part 8 of the metric $\mathfrak{m}_{T*}$ and ω_{T*}. With L_1 chosen, assume that $L > 4L_1$. The metric $\mathfrak{m}_{T*}$ where $s \in [-\frac{1}{2}L+4, -\frac{1}{2}L+5]$ is the same as the $s \in [-\frac{1}{2}L+4, -\frac{1}{2}L+5]$ version of the metric from Part 8; and ω_{T*} on this same part of X is the $s \in [-\frac{1}{2}L+4, -\frac{1}{2}L+5]$ version of Part 8's 2-form w_X. This understood, the desired metric for X is taken to be $\mathfrak{m}_{T*}$ where $s \geq -\frac{1}{2}L+4$, and the 2-form w_X is taken to be ω_{T*} on this same part of X. Here, υ_X is set to be the s-independent 1-form $\upsilon_\diamond$; and the bounds in items 4b) and 5c) of (3.13) are verified by choosing the parameter m to be sufficiently small, as directed in Part 2 above. □

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